Feynman Integral, Knot Invariant and Degree Theory of Maps

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Abstract

The universal Vassiliev invariant from the perturbative Chern-Simons theory is actually a knot invariant without any correction term. The anomaly considered by Bott and Taubes is proved to be zero.

Contents

§0 Introduction

Following the works of Bar-Natan [3], Bott-Taubes [5] and Altschuler-Freidel [1], we know how to use Feynman diagrams and their associated integral to get the beautiful and natural Universal Vassiliev Invariant in the infinite dimensional algebra of chord diagram (or Feynman diagram). But, there still have some defect. A correction term coming from the integrals over the spaces of totally concentrated Feynman diagrams should be considered. The main purpose of this article is to show that the correction term is equal to zero.

0.1

Roughly speaking, a Feynman diagram is a type of graph in $\mathbb{R}^3$ with partial vertices staying on a knot K. If Γ denotes a Feynman diagram, the Feynman integral $I(\Gamma, K)$ is a measure for the configuration space $C(\Gamma, K)$ of graphs, equiv. to Γ, on K. $|\Gamma|$ denotes the number of automorphisms of Γ. ($|\Gamma|$ is also equal to the multiplicity of Γ appearing in $C(\Gamma, K)$.) The Universal Vassiliev invariants, proposed by Bar-Natan [3], also by Kontsevich [8], can be written as

$$Z(K) = \sum_{\Gamma} \frac{I(\Gamma, K)}{|\Gamma|}[\Gamma],$$

(the summation is over equivalence classes of Feynman diagrams)

where K is a knot and $[\Gamma]$ denotes the corresponding element of Γ in the algebra of (chord) diagram. Atschuler and Freidel [1] consider the following correction term:

$$\alpha = \frac{1}{2} \sum_{\Gamma} \frac{f_\Gamma}{|\Gamma|}[\Gamma],$$

which is quite similar to the above $Z(K)$ except the integral part f_Γ. f_Γ, invented by Bott and Taubes [5], is a measure for the "universal space"

$W(\Gamma)$ of totally collapsed Feynman diagrams which is equivalent to Γ. The "universal space" $W(\Gamma)$ has the same dimension as $C(\Gamma, K)$, and f_Γ is also a Feynman integral similar to $I(\Gamma, K)$. The interesting thing is that the correction term α is independent of K and what are the values of α and f_Γ, for each Γ. Altschuler and Freidel showed that, when Γ is not connected as a graph or the order of Γ is even, f_Γ is always zero.

In this article, we propose a method to show that the sequence $\alpha = \frac{1}{2}\sum\limits_{\Gamma}\frac{f_\Gamma}{|\Gamma|}[\Gamma]$ is equal to zero in the algebra of diagrams. A weight system ω is a integral value function for the diagrams such that $\sum\limits_{\Gamma}\omega(\Gamma)C(\Gamma, K)$ forms a "cycle in the homology theory". Thus, to show that $\alpha = 0$ in the algebra of diagram, it is enough to show that $\omega(\alpha) = \frac{1}{2}\sum\limits_{\Gamma}\frac{f_\Gamma}{|\Gamma|}\omega(\Gamma)$ is zero, for any weight system ω.

As above, we use $W(\Gamma)$ denote the space of totally collapsed Feynman diagrams equivalent to Γ.

By the result of Bott and Taubes, $\sum\limits_{\Gamma}\omega(\Gamma)C(\Gamma, K)$ can not form a cycle ($W(\Gamma)$ is related to the anomalous boundary of $C(\Gamma, K)$). But, when we consider the analogue for $W(\Gamma)$, $\sum\limits_{\Gamma}\omega(\Gamma)W(\Gamma)$ does form a cycle under suitable interpretation. And the value $\omega(\alpha) = \frac{1}{2}\Sigma\frac{f_\Gamma}{|\Gamma|}\omega(\Gamma)$ is equal to the degree of canonical map from $W(\Gamma)$ to a special space B_k, which looks like the classifying space of diagrams with k edges.

Furthermore, $W(\Gamma)$ has a natural $SO(3)$-action and it is a fibre bundle over S^2 with fibre $W_0(\Gamma)$, a S^1-space. Thus, $W(\Gamma) = W_0(\Gamma) \underset{S^1}{\times} SO(3)$, and the canonical map $\Phi_0 : W_0(\Gamma) \longrightarrow B_k$ determines $\Phi : W(\Gamma) \longrightarrow B_k$ completely.

Now, we shall move the problem to $W_0(\Gamma)$ and $\Sigma\omega(\Gamma)W_0(\Gamma)$. The S^1-action on $W_0(\Gamma)$ is semifree, that is, the isotropic groups are trivial or the whole S^1. Let $H(\Gamma)$ denote the fixed point set of the S^1-action on $W_0(\Gamma)$

and $D(\Gamma)$ be the normal vector bundle of $H(\Gamma)$ in $W_0(\Gamma)$. Then $\Sigma\omega(\Gamma)D(\Gamma)$ also forms a vector bundle over $\Sigma\omega(\Gamma)H(\Gamma)$ and is the normal bundle of $\Sigma\omega(\Gamma)H(\Gamma)$ in $\Sigma\omega(\Gamma)W_0(\Gamma)$. And, the interesting facts are the following: (i) the "degree" of $\Sigma\omega(\Gamma)W(\Gamma)$ is equal to a Chern number of the normal bundle $\Sigma\omega(\Gamma)D(\Gamma)$, (ii) all the possible normal vector bundles has structure group isomorphic to the semi-direct product of $\bigoplus_k \mathbf{Z_2}$ and the symmetry group Σ_k. This concludes that the possible degrees are all zero.

Remark : In the above consideration, we may restrict the diagrams Γ with the same order n. But, under the order restriction, $W(\Gamma)$ may also have different dimensions. Thus, we should consider the stable one $W(\Gamma)^{(r)} = W(\Gamma) \times (\prod_r \mathbb{RP}^2)$, and arrange them into spaces with the same dimension.

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In the remaining sections of the introduction, we talk about the tools and strategies using in the article.

0.2 Knot graph and its configuration space:

Suppose K is a oriented knot or a oriented simple curve. A knot graph Γ on K is a finite 1-dimensional simplicial complex with two kinds of vertices in $V_0(\Gamma)$ and $V_1(\Gamma)$, elements in $V_0(\Gamma)$ are called base points which are always stay in K, and elements in $V_1(\Gamma)$ are called inner vertices which are only assumed to be points of $\mathbb{R}^3$ (could be in K). $V(\Gamma) = V_0(\Gamma) \cup V_1(\Gamma)$ is total vertice set. The edge set $E(\Gamma)$ is a finite set of unordered pairs of distinct points in $V(\Gamma)$, that is, for $e \in E(\Gamma)$, $e = \{v, w\}$ with $v \neq w$ in $V(\Gamma)$. For convenience, choose a linear on K such that the increasing order is the same as the given orientation.

Suppose Γ_1 is a knot graph on K_1 and Γ_2 is a knot graph on K_2. A simplicial map $g : \Gamma_1 \longrightarrow \Gamma_2$ is said to be an equivalence of knot graphs, if g

is an isomorphism of simplicial complexes, $g(V_0(\Gamma_1)) = V_0(\Gamma_2)$, $g(V_1(\Gamma_1)) = V_1(\Gamma_2)$, $g(E(\Gamma_1)) = E(\Gamma_2)$, and g preserves the linear orders of $V_0(\Gamma_1)$ and $V_0(\Gamma_2)$, which are inherited from the linear orders of K_1 and K_2, respectively.

Let $\overline{C}(\Gamma, K) = \{\Gamma' : \Gamma'$ is a knot graph on K and Γ' is equivalent to Γ by an equivalence $g\}$, it is called the configuration space of Γ (over K).

The interesting thing for knot graph and its configuration space is to find a measure for the configuration spaces, i.e. integral on $\overline{C}(\Gamma, K)$ and a method to collect the knot graphs $\Gamma_1, \Gamma_2, \cdots$ such that their measure $\sum_i I(\overline{C}(\Gamma_i, K))$ constitutes a knot invariant.

For convenience, consider $C(\Gamma, K) = \{(g, \Gamma') : \Gamma'$ is a knot graph on K and $g : \Gamma \longrightarrow \Gamma'$ is an equivalence $\}$, it is a covering space of $\overline{C}(\Gamma, K)$. Sometimes, for simplicity, just denote it by $C(\Gamma)$.

Whenever Γ is a knot graph, there is a natural map $\phi_\Gamma : E(\Gamma) \longrightarrow \mathbb{RP}^2$ $= \{$ lines in $\mathbb{R}^3$ through the origin $\}$, defined by: $e = \{v, w\} \in E(\Gamma)$, $\phi_\Gamma(e)$ is the line passing through $(v - w)$ and the origin. If Γ has k edges, then the image of ϕ_Γ determines an element in $\prod_k \mathbb{RP}^2/\Sigma_k$, the k-fold symmetric product of $\mathbb{RP}^2$. Let $B_k = \prod_k \mathbb{RP}^2/\Sigma_k$, it looks like a classifying space of knot graphs. Then, there is a canonical smooth map $\Phi : C(\Gamma, K) \longrightarrow B_k$, $\Phi(\Gamma') = Im(\phi_{\Gamma'})$. (of course, also for $\overline{C}(\Gamma, K)$). $C(\Gamma, K)$ is not compact. But, it is very fortunate that there is a compactification (Fulton-MacPherson [6]) such that Φ can be smoothly extended to this compactification and the codimension 1 boundary is the union of $C(\Gamma; A)$, where A run over all the subsets of $V(\Gamma)$, containing at least two points. (details see section 1.2)

Definition : (i) $B_k = \prod_k \mathbb{RP}^2/\Sigma_k$, Σ_k, the symmetry group acting on $\prod_k \mathbb{RP}^2$ by permuting the coordinates.

(ii) (X, f) is said to be a B_k-space, if X is space and $f : X \longrightarrow B_k$ is continuous.

(iii) $P(k) = \prod_k \mathbb{RP}^2$.

(iv) (X, f) is said to be a $P(k)$-space, if X is a space and $f : X \longrightarrow P(k)$ is continuous.

Examples: $C(\Gamma)$ has a canonical B_k-structure $\Phi : C(\Gamma) \longrightarrow B_k$. When we choose an order for the edges in Γ, we have a $P(k)$-structure $\Psi : C(\Gamma) \longrightarrow P(k)$ in the obvious way, but there are $k!$ different $P(k)$-structures dependent on the $k!$ different choices of orders. $\overline{C}(\Gamma)$ also has a canonical B_k-structure, but may not have a $P(k)$-structure. The boundaries $C(\Gamma; A)$ are similar to the situation for $C(\Gamma)$.

To get an a priori measure for $(C(\Gamma), \Phi)$, we need to introduce the line bundles for the B_k-apces and the associated local coefficient for these B_k-spaces.

Let L'_k be the orientation line bundle over $P(k)$ (thus L'_k is non-orientable) and L_k be the line bundle over B_k "push down" from L'_k, that is, $L_k = L'_k/\Sigma_k$.

For any B_k-space (X, f), let $L(X)$ denote the pull-back $f^{\#}(L_k)$ of L_k by f. (X, f) and (X', f') are two B_k-space, $\tau : X \longrightarrow X'$ is said to be a B_k-map, if $f' \circ \tau = f$. A B_k-map $\tau : X \longrightarrow X'$ determines uniquely a bundle map $\overline{\tau} : L(X) \longrightarrow L(X')$ such that $\overline{f}' \circ \overline{\tau} = \overline{f}$, where $\overline{f} : L(X) \longrightarrow L_k$ and $\overline{f}' : L(X') \longrightarrow L_k$ are the bundle maps covering f and f' respectively.

Definition : Suppose X is a manifold. A B_k-orientation for a B_k-manifold (X, f) is an orientation for the total space of the line bundle $L(X)$.

Let $\mathcal{O}_k$ denote the orientation sheaf of L_k, then $f^{\#}\mathcal{O}_k$ is the orientation sheaf of $L(X)$ over X. $H_{2k}(B_k, \mathcal{O}_k) \simeq \mathbf{Z}$ and $f_* : H_{2k}(X, f^{\#}\mathcal{O}_k) \longrightarrow H_{2k}(B_k, \mathcal{O}_k)$ provides a most natural measure for the $2k$-dimensional cycles in X.

In Chapter 1, we shall use the configuration spaces constructing a big "knot invariant" B_k-space, and the weight systems of Vassiliev invariant will

provide "tautological" cycles in this B_k–space and the associated degrees from the homology theory are the knot invariants.

The "troubles" of "knot invariant" B_k-space will be solved by introducing another big B_k-space $\mathcal{W}$ whose degrees are always trivial. (B_k-space $\mathcal{W}$ is defined in Chapter 3.)

0.3 Construction of "knot invariant" B_k-space:

The main tool of this construction is the identification map for the boundary $C(\Gamma; A)$ of configuration space $C(\Gamma)$.

For any graph Γ and a subset A of vertice set $V(\Gamma)$, $A(\Gamma)$ denotes the subgraph of Γ inside A. Precisely, the vertice set of $A(\Gamma)$ is A and the edge set $E(A(\Gamma))$ is equal to $\{e = \{v, w\} \in E(\Gamma) : v, w \in A\}$. The codimension 1 boundary $C(\Gamma; A)$ is a fibre bundle over $C(\Gamma/A)$ with fibre $D(A(\Gamma), x)$.(details see section 1.2 and 1.3)

The identification maps for $C(\Gamma; A)$ have two kinds:

(i) First kind: When $|A| \geq 3$ and $A(\Gamma)$ has one of the following three type of vertice: free vertice, univalent inner vertice or bivalent inner vertice, there is a smooth map:

$$\tau : C(\Gamma; A) \longrightarrow C(\Gamma; A).$$

(ii) Second kind: When $|A| = 2$ (that is, A has exactly two vertices), there is a smooth map:

$$\tau : C(\Gamma; A) \longrightarrow C(\Gamma/A) \times P(A),$$

where

$$P(A) = \begin{cases} \mathbb{RP}^2, & \text{if } A(\Gamma) \text{ has one edge,} \\ \{\text{one point}\}, & \text{if } A(\Gamma) \text{ has no edge.} \end{cases}$$

It is easy to see that, when X is a B_k-space, $X^{(r)} = X \times B_r$ is a B_{k+r}-space. And the identification map can be extended directly to $\tau : C(\Gamma; A)^{(r)} \longrightarrow C(\Gamma; A)^{(r)}$ (first kind) and $\tau : C(\Gamma; A)^{(r)} \longrightarrow C(\Gamma/A)^{(r+e(A))}$ (second kind), where $e(A)$ denote the number of edge in $A(\Gamma)$.

Definition : Suppose Γ is a knot graph.

$$\text{order } (\Gamma) = |E(\Gamma)| - |V_1(\Gamma)|.$$

Now, we restrict to the knot graphs of order n. Let $\overline{\mathcal{S}}(n)$ be the disjoint union of $C(\Gamma)^{(3n-k)}$ (k is the edge number of Γ), for all Γ with order n and having no free vertice, univalent inner vertice and bivalent inner vertice. Furthermore, let $\mathcal{S}(n)$ denote the quotient space of $\overline{\mathcal{S}}(n)$ by identifying α with $\tau(\alpha)$, for all possible identification maps τ of first kind and second kind, and for all α in the corresponding boundary $C(\Gamma; A)$. Because the identification rule to identify some points in $\overline{\mathcal{S}}(n)$ which have the same image in B_{3n} (note: $C(\Gamma)^{(3n-k)}$ is a B_{3n}-space), $\mathcal{S}(n)$ is also a natural B_{3n}-space. Let $\Phi : \mathcal{S}(n) \longrightarrow B_{3n}$ also denote the structure map. Then, $\Phi_* : H_{6n}(\mathcal{S}(n), \Phi^{\#}(\mathcal{O}_{3n})) \longrightarrow H_{6n}(B_{3n}, \mathcal{O}_{3n})$ is supposed to give a lot of knot invariant. Unfortunately, there is none. Originally, for any weight system ω with integral coefficient, one expects that $\Sigma\omega(\Gamma)C(\Gamma)$ is a tautological cycle in $(\mathcal{S}(n), \Phi^{\#}(\mathcal{O}_{3n}))$ and $\Phi_*(\Sigma\omega(\Gamma)C(\Gamma))$ will provide a knot invariant for K. But, Bott and Taubes [5] showed that $\Sigma\omega(\Gamma)C(\Gamma)$ has extra boundary $C(\Gamma; A)$, $A = V(\Gamma)$, the total vertice set, which is not considered in the above identifications. We find that $\Phi_*(\Sigma\omega(\Gamma)C(\Gamma))$ actually could also be a cycle in $(B_{3n}, \mathcal{O}_{3n})$. But, it is hard to find further "tautological" identifications in the space $\mathcal{S}(n)$. Thus, we propose to consider additional configuration spaces of the totally collapsed knot graphs $W(\Gamma, K)$ such that $\Sigma\omega(C(\Gamma, K) + W(\Gamma, K))$ could be a cycle in some B_{3n}-space which contains $\mathcal{S}(n)$.

But, the trouble of $W(\Gamma, K)$ is that $W(\Gamma, K)$ depends on a domain $D(K)$ in S^2. The boundary of $D(K)$ is the closed curve of unit tangent of K. Such

a domain has two choice $D_1(K)$ and $D_2(K)$, $D_1(K) \cup D_2(K) = S^2$. And, $W(\Gamma, K)$ has also two choice: $W(\Gamma, D_1(K))$ and $W(\Gamma, D_2(K))$. If one show that $\Sigma\omega(\Gamma)(W(\Gamma, D_1(K)) + W(\Gamma, D_2(K)) = \Sigma\omega(\Gamma)W(\Gamma, S^2)$ always gets zero value in $H_{6n}(B_{3n}, \mathcal{O}_{3n})$ (under Φ_*). Then, the Φ_*-value of $\Sigma\omega(\Gamma)(C(\Gamma, K) + W(\Gamma, K))$ for K is a knot invariant.

(Sorry, the above argument is quite rough, the correct form of tautological cycle might be much complicated.)

In the following, we write $W(\Gamma)$ for $W(\Gamma, S^2) = W(\Gamma, D_1(K)) \cup W(\Gamma, D_2\ (K))$, which is independent of K.

Now, we apply the same trick for $W(\Gamma)$, instead of $C(\Gamma)$, ($\dim C(\Gamma) = \dim W(\Gamma)$). And, for some technical reason, we consider the $P(k)$-structure of $W(\Gamma)$, instead of B_k-structure.

0.4 The $P(3n)$-space $\mathcal{W}_0(n)$ and $\mathcal{W}(n)$:

Instead of a knot, we consider the line l_0 of z-axis and the configuration space $C(\Gamma, l_0)$ of knot graphs on l_0, which is equivalent to Γ. When we translate a knot graph on l_0 in the line direction or dilate a knot graph, the knot graph does not change its image in $P(k)$ or B_k. Thus, $W_0(\Gamma) = C(\Gamma, l_0)/(\text{translation and dilation relation})$ is also a $P(k)$-space. But, now, we should consider all $P(k)$-structures by interchanging the order of edges, that is, $\Sigma_k \cdot \Psi$, where $\Psi : W_0(\Gamma) \longrightarrow P(k)$ is some canonical $P(k)$-structure.

Let $\overline{\mathcal{W}}_0(n)$ be the disjoint union of all $W_0(\Gamma) \times P(3n-k) \times \Sigma_{3n} \cdot \Psi$ and $\mathcal{W}_0(n)$ be the quotient space of $\overline{\mathcal{W}}_0(n)$ by the identification maps with a reasonable choice of $P(3n)$-structure and the extended translation and dilation relations. (see section 3.1.)

Explanation: The extended translation and dilation relation is defined on $W_0(\Gamma)$, for the splittable knot graph Γ. The purpose is to reduce the dimension of $W_0(\Gamma)$ by one or two such that we may forget the splittable

knot graphs in $\mathcal{W}_0(n)$. A knot graph is said to be splittable, if it is a union of two subgraphs whose intersection is empty or exactly a base point.

0.4.1 S^1 action on the spaces $W_0(\Gamma), \mathcal{W}_0(n)$

Identify S^1 as the $SO(2)$-subgroup of $SO(3)$, which contains the rotations around the z-axis. Then S^1 naturally acts on $C(\Gamma, l_0)$, and hence on $W_0(\Gamma)$. The identification maps in the boundaries of $W_0(\Gamma)$ are all S^1-equivariant. Thus, the S^1-action is also well-defined on $\mathcal{W}_0(n)$.

l_0 is oriented by the direction $(0,0,1)$. For any $x \in S^2$, we have the line $l_x = \{tx : t \in \mathbb{R}\}$ with orientation x. We may define $W_x(\Gamma)$ and $\mathcal{W}_x(n)$ in the same way. Then, $W(\Gamma) = \bigcup\limits_{x \in S^2} W_x(\Gamma)$ with suitable topology, it is a fibre bundle over S^2 with fibre $W_0(\Gamma)$, and $\mathcal{W}(n) = \bigcup\limits_{x \in S^2} \mathcal{W}_x(n)$, also. Because $SO(3)$ acts on S^2 transitively, we may write $W(\Gamma) = SO(3) \underset{S^1}{\times} W_0(\Gamma)$ and $\mathcal{W}(n) = SO(3) \underset{S^1}{\times} \mathcal{W}_0(n)$, where the topology of $W(\Gamma)$ and $\mathcal{W}(n)$ is obvious, instead of the loose union. Actually, $SO(3)$ has a natural action on $\bigcup\limits_{x \in S^2} W_x(\Gamma)$, which is the same as the $SO(3)$-action from $SO(3) \underset{S^1}{\times} W_0(\Gamma)$ (also for $\mathcal{W}(n)$). $SO(3)$ also acts on $\mathbb{RP}^2$ and $P(k)$. The $P(k)$-structures of $W(\Gamma)$ and $\mathcal{W}(n)$ are also $SO(3)$-equivariant. Thus, one might expect that the S^1-equivariant $P(3n)$-structure of $\mathcal{W}_0(n)$ could control the $SO(3)$-equivariant $P(3n)$-structure of $\mathcal{W}(n)$ nicely.

0.4.2 Orbit space of S^1-action:

Consider the orbit spaces $\mathcal{W}_0(n)/S^1$ and $P(3n)/S^1$ and the map $\overline{\Psi} : \mathcal{W}_0(n)/S^1 \longrightarrow P(3n)/S^1$ induced by $P(3n)$-structure $\Psi : \mathcal{W}_0(n) \longrightarrow P(3n)$. We find that the homomorphism $\overline{\Psi}_* : H_{6n-3}(\mathcal{W}_0(n)/S^1) \longrightarrow H_{6n-3}(P(3n)/S^1)$ has a good relation with the degree homomorphisms from $H_{6n}(\mathcal{W}(n))$ to $H_{6n}(P(3n))$, (all the homologies are with suitable local coefficients). Even better, when we restrict the map $\overline{\Psi}$ to any neighborhood of the fixed point set

of $\mathcal{W}_0(n)$ and $P(3n)$, we can still control the original degree homomorphism of $\mathcal{W}(n)$.

$P(3n)$ has only one fixed point ξ_0, we may choose the canonical Euclidean space $\mathbb{R}^{6n} = \mathbb{C}^{3n}$ as the neighborhood of ξ_0 and its inverse image in $\mathcal{W}_0(n)$ is a $2n$-dimensional complex vector $\mathcal{D}(n)$ over the $(2n-2)$-dimensional fixed point set $\mathcal{H}(n)$ of the S^1-action on $\mathcal{W}_0(n)$.

Now, consider the restriction $\overline{\overline{\Psi}}$ of $\overline{\Psi}$ to $\mathcal{D}(n) - \mathcal{H}(n)/S^1$, we have a map from the S^1-orbit space of the associated spherical bundle to $\mathbb{C}^{3n} - \{0\}/S^1$, which is homotopically equivalent to $\mathbb{CP}^{3n-1}$. Its homology homomorphism is just a stable characteristic class of the complex vector bundle $\mathcal{D}(n)$ over $\mathcal{H}(n)$.

Up to now, we have the spaces in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}(n) \arrow[r, hook] & \mathcal{D}(n) \arrow[r, "\Psi"] \arrow[d] & \mathbb{C}^{3n} \arrow[d] \\
 & \mathcal{W}_0(n) \arrow[r, "\Psi"] \arrow[d] & P(3n) \\
 & \mathcal{W}(n) \arrow[ur, "\Psi"] &
\end{tikzcd}$$

(The vertical maps all are inclusions.)

We also have the orbit spaces which form a commutative diagram as follows:

$$
\begin{array}{ccccc}
\mathcal{D}(n) - \mathcal{H}(n)/S^1 & \xrightarrow{\overline{\overline{\Psi}}} & \mathbb{C}^{3n} - 0/S^1 & \simeq & \mathbb{CP}^{3n-1} \\
\downarrow & & \downarrow & & \\
\mathcal{D}(n)/S^1 & \longrightarrow & \mathbb{C}^{3n}/S^1 & & \\
\downarrow & & \downarrow & & \\
\mathcal{W}_0(n)/S^1 & \xrightarrow{\overline{\Psi}} & P(3n)/S^1 & &
\end{array}
$$

Now, we can state the Theorems which are needed for the proof of our main result.

Theorem A: If $\overline{\overline{\Psi}}_* : H_{6n-4}(\mathcal{D}(n) - \mathcal{H}(n)/S^1) \longrightarrow H_{6n-4}(\mathbb{CP}^{3n-1})$ is a zero-homomorphism, then the degree homomorphism $\Psi_* : H_{6n}(\mathcal{W}(n)) \longrightarrow H_{6n}(P(3n))$ (with suitable local coefficient) is also a zero homomorphism.

Theorem B: The stable characteristic classes of $\mathcal{D}(n)$ over $\mathcal{H}(n)$ are all zero. Thus, $\overline{\overline{\Psi}}_* = 0$, in any dimensions.

The proof of Theorem A has two steps:

(i) $\overline{\Psi}_* : H_{6n-3}(\mathcal{W}_0(n)/S^1) \longrightarrow H_{6n-3}(P(3n)/S^1)$ is 0-homomorphism $\Longrightarrow$ degree homo. $(\Psi_*)_{6n}$ is 0.

(ii) "$(\overline{\overline{\Psi}}_*)_{6n-4} = 0$" $\Longrightarrow$ "$(\overline{\Psi}_*)_{6n-3} = 0$"

Step (i) is a result of functorial property of $SO(3) \underset{S^1}{\times} (\cdot)$.

Step (ii) depends on the following

Proposition: $H_{6n-3}(P(k) - \xi_0/S^1) = 0$ or $H_{6n-3}(\prod_{3n} S^2 - F/S^1) = 0$, where F is the fixed point set of S^1-action on $\prod_{3n} S^2$.

Both results in the proposition can prove Step (ii), we actually prove the second. (see section 4.4)

And the proof of Theorem B is to show that the vector bundle $\mathcal{D}(n)$ over $\mathcal{H}(n)$ has finite structure group.

0.5 Conclusion

Suppose ω is a weight system for the Vassiliev invariant of order n. Then $\sum\limits_{\Gamma} \omega(\Gamma) W(\Gamma) \times P(3n-k) \times \Sigma_{3n} \cdot \Psi$ forms a $6n$-dimensional cycle in $(\mathcal{W}(n), \Psi^*(\mathcal{O}_{3n}))$ and its image in $H_{6n}(P(3n), \mathcal{O}_{3n})$ is equal to a non-zero multiple of $\sum\limits_{\Gamma} \frac{f_\Gamma}{|\Gamma|}\omega(\Gamma)$. Thus, $\sum\limits_{\Gamma} \frac{f_\Gamma}{|\Gamma|}\omega(\Gamma) = 0$, for any weight system ω, and hence, $\sum\limits_{\Gamma} \frac{f_\Gamma}{|\Gamma|}[\Gamma] = 0$ in the algebra of chord diagram.

§1. Degree theory and knot invariant

In this part, we define the knot graphs and their configuration spaces, discuss the boundaries and see how to identify the boundaries of different configuration spaces to get the knot invariants.

1.1 Knot graph and Configuration space

1.1.1 Knot graph

A graph Γ is an abstract 1-dimensional finite simplicial complex with vertices in $V(\Gamma)$ and edges in $E(\Gamma)$. For $e \in E(\Gamma)$, e is an unordered pair of distinct vertices, that is, $e = \{v, w\}$, $v \neq w$ in $V(\Gamma)$.

$\kappa : S^1 \longrightarrow \mathbb{R}^3$ is an embedding, $K = \kappa(S^1)$. Fix the linear order for the points on K as follows:

$$\kappa(1) < \kappa(e^{i\theta}) < \kappa(e^{i\theta'}), \text{ for } 0 < \theta < \theta' < 2\pi.$$

Definition (1.1): A graph Γ is said to be a knot graph on K, if $V(\Gamma) \subset \mathbb{R}^3$ and $V(\Gamma)$ is disjoint union of $V_0(\Gamma)$ and $V_1(\Gamma)$ such that $V_0(\Gamma)$ is a subset of K. The vertices in $V_0(\Gamma)$ are called base points. The vertices in $V_1(\Gamma)$ are called inner vertices which are only assumed to be points of $\mathbb{R}^3$. Any two vertices are distinct points of $\mathbb{R}^3$.

$\mathbb{RP}^2$ denotes the space of straight lines in $\mathbb{R}^3$ passing through the origin, it is also diffeomorphic to S^2 quotiented by the $\mathbf{Z_2}$–action: $-1 \cdot x = -x$. Any non-zero vector x of $\mathbb{R}^3$ determines an element $[x] = \{tx, t \in \mathbb{R}\}$ of $\mathbb{RP}^2$.

The interest of knot graph Γ and $\mathbb{RP}^2$ is the following map for the edges: $\phi_\Gamma : E(\Gamma) \longrightarrow \mathbb{RP}^2$, for the $e = \{v, w\} \in E(\Gamma)$, $\phi_\Gamma(e) = [v-w]$. If X is a space of knot graphs, we have a canonical map associated with X, $\Phi : X \longrightarrow \{$ finite subsets of $\mathbb{RP}^2\}$, $\Phi(\Gamma) = \{\phi_\Gamma(e) : e \in E(\Gamma)\}$, for $\Gamma \in X$. When all the

graphs in X have the same number of edges, say k, then Φ is a well-defined map from X to the k-fold symmetric product of $\mathbb{RP}^2$, $\prod_k \mathbb{RP}^2/\Sigma_k$, where Σ_k is the symmetry group of $\{1, 2, \cdots, k\}$ and Σ_k acts on $\prod_k \mathbb{RP}^2$ by permuting the coordinates.

1.1.2 Equivalence of knot graphs

Suppose K, K' are knots or simple curves with a natural linear order for their points.

Definition (1.2): Assume Γ is knot graph on K and Γ' is a knot graph on K'. A simplicial map $g : \Gamma \longrightarrow \Gamma'$ is said to be an equivalence of knot graphs, if g is an isomorphism of simplicial complexes and the restriction of g to $V_0(\Gamma)$ is an order preserving bijection from $V_0(\Gamma)$ onto $V_0(\Gamma')$. (Thus, g sends base points to base point and sends inner vertices to inner vertices.)

For any knot graph Γ, the base points of Γ have a linear order which is inherited from the linear order of points in the knot or simple curve. The equivalence g must preserve the linear orders of base points. Furthermore, g is a bijection of $E(\Gamma)$ and $E(\Gamma')$.

Two equivalent knot graphs may be not on the same knot. Thus, it is convenient to consider a equivalence class of knot graphs or a single knot graph as an abstract 1-dimensional simplicial complex with two type of vertices: one is the base points with a linear order and one is the inner vertices.

1.1.3 Configuration space of knot graphs

Definition (1.3): Suppose Γ is a knot graph (on some knot or simple curve with linear order) and K is a knot or a simple curve in $\mathbb{R}^3$. $C(\Gamma, K)$ denotes the set of all pairs (g, Γ'), Γ' is a knot graph on K and g is equivalence from Γ to Γ'.

An alternating definition of configuration space is the following: $\overline{C}(\Gamma, K) = \{\Gamma' : \Gamma'$ is a knot graph on K and Γ' is equivalence to $\Gamma\}$. It is easy to see that $C(\Gamma, K)$ is a covering space of $\overline{C}(\Gamma, K)$, and the number of elements in the fibre is the same as the number of automorphisms of Γ. $|\Gamma|$ will be used to denote the order of automorphism group $\{g : \Gamma \longrightarrow \Gamma$ equivalence $\}$ of Γ.

If Γ has m base points and s inner vertices, then the dimension of $C(\Gamma, K)$ is $m + 3s$.

If Γ has k edges $e_1, e_2, \cdots, e_k$. Let $B_k = \prod_k \mathbb{RP}^2/\Sigma_k$. Φ denotes the canonical map from $C(\Gamma, K)$ to B_k, that is, $\Phi\ (g, \Gamma') = Im\ (\phi_{\Gamma'})$. Ordinarily, $\Phi(C(\Gamma, K))$ also has dimension $m+3s$ in B_k. But, sometimes, it is degenerate. For example, when K is straight line in $\mathbb{R}^3$, $\dim \Phi(C(\Gamma, K)) \leq m + 3s - 2$. The reason is that when a knot graph on a straight line translates on the line or dilates all give the same image in B_k. In this article, the knot invariant will be a correct measure of $\Phi(C(\Gamma_i, K))$ or $\Phi(\overline{C}(\Gamma_i, K))$, $i = 1, 2, \cdots, l$, for some Γ_i, $i = 1, 2, \cdots, l$. Thus, one of important work is to find out some principles or some rules when or where there are two distinct knot graphs or a bunch of knot graphs which have the same image in B_k. Then, we can make the identifications at first and use them or disuse them for the purpose of knot invariant.

Example (1.4): Suppose Γ has a univaliant inner vertice v, that is, some edge $e = \{v, w\}$ is only one edge containing v as the end point. Then, each $(g, g(\Gamma))$ in $C(\Gamma, K)$ has the same image in B_k as the bunch of elements in $\{g_t, g_t(\Gamma)), t \neq 0\}$, where $g_t(v) = g(w) + t(g(v) - g(w))$ and $g_t(v') = g(v')$, for $v' \neq v$. Thus, we may identify $\{g_t, g_t(\Gamma))\}$ to a point for each $(g, g(\Gamma))$ in $C(\Gamma, K))$, and the new space has less 1 dimension. In the following, we shall find more and more identification rules to make up spaces for knot invariant.

The most important identification rules will happen in the boundaries of configuration spaces $C(\Gamma, K)$.

1.2. Boundary of Configuration Spaces

1.2.1. Description of Boundaries

In the remaining part of this chapter, K will be fixed and $C(\Gamma, K)$ will be denoted simply by $C(\Gamma)$, for any knot graph Γ.

$C(\Gamma)$ is, by no mean, compact, but there is a compactification (Fulton and MacPherson [6]) for the configuration spaces such that $\Phi : C(\Gamma) \longrightarrow B_k$ can be extended continuously to the compactification of $C(\Gamma)$. For convenience, we assume that $C(\Gamma)$ has been substituted by its Fulton-MacPherson compactification.

(1.5) (Fulton-MacPherson) The codimension 1 boundary of $C(\Gamma)$ is the union of $C(\Gamma; A)$, for any A which is a subset of the vertice set $V(\Gamma)$ and contains at least two vertices. $C(\Gamma; A)$ is a fibre bundle over $C(\Gamma/A)$ with fibres $D(A(\Gamma), x)$, where Γ/A and $D(A(\Gamma), x)$ are defined in (1.6) and (1.8).

(1.6) Γ/A denotes the knot graph with vertice set $V(\Gamma/A) = (V(\Gamma)-A) \cup \{a\}$, $a \notin V(\Gamma)$, and edge set $E(\Gamma/A) = \{e = \{v, w\} : v, w \notin A,\ e \in E(\Gamma)\} \cup \{e = \{v, a\} : v \notin A$ and there exists $w \in A$ such that $\{v, w\} \in E(\Gamma)\}$. The set of base points in Γ/A is the following:

$$V_0(\Gamma/A) = \begin{cases} V_0(\Gamma), & \text{if } A \cap V_0(\Gamma) = \phi, \\ \{a\} \cup V_0(\Gamma) - A, & \text{if } A \cap V_0(\Gamma) \neq \phi. \end{cases}$$

Because the new vertice a replaces the vertices in A, the linear order on $V_0(\Gamma/A)$ can be defined in the obvious way, under the Assumption(1.7) below.

Assumption (1.7): $A \cap V_0(\Gamma)$ is an interval of the linear order set $V_0(\Gamma)$, that is, $\{v \in V_0(\Gamma) : v_1 \leq v \leq v_2\}$, for some v_1, v_2 in $V_0(\Gamma)$.

(1.8) Description of $D(A(\Gamma), x)$

Suppose $(g, g(\Gamma/A))$ is an element in $C(\Gamma/A)$ and x is the point $g(a)$. Then $D(A(\Gamma), x)$ is the fibre of $C(\Gamma; A)$ over $(g, g(\Gamma/A))$.

(i) $A(\Gamma)$ is the graph with vertice set $V(A(\Gamma)) = A$ and edge set $E(A(\Gamma)) = \{e = \{v, w\} \in E(\Gamma) : v$ and w are in $A\}$. When $A \cap V_0(\Gamma)$ is non-empty, $A(\Gamma)$ is a knot graph with set of base points $V_0(A(\Gamma)) = A \cap V_0(\Gamma)$. When $A \cap V_0(\Gamma)$ is empty, $A(\Gamma)$ is a graph without base point. Thus, whether A contains base points or not, the situation is quite different.

(ii) (Case 1): $A \cap V_0(\Gamma)$ is non-empty.

In this case, a is a base point. Thus, $x = g(a)$ is a point on K and $D(A(\Gamma), x)$ is the set of "infinitesimal" knot graphs concentrated at x. Precisely, let $l(x)$ denote the tangent line of K at x, $D(A(\Gamma), x)$ is the configuration space $C(A(\Gamma), l(x))$ quotiented by the translation and dilation relation: $(g_1, g_1(A(\Gamma)))$ and $(g_2, g_2(A(\Gamma)))$ are two element in $C(A(\Gamma), l(x))$, $g_1 \sim g_2$ if there exist $\lambda > 0$ and y_0 in $\mathbb{R}^3$ such that $g_1(v) = y_0 + \lambda g_2(v)$ for all $v \in V(A(\Gamma)) = A$. This definition is good for any kind of graph with or without base point. But, in this case, y_0 could only be a vector parallel to $l(x)$. Thus, $\dim D(A(\Gamma), x) = \dim C(A(\Gamma), l(x)) - 2$, and $\dim C(\Gamma; A) = \dim C(\Gamma/A) + \dim D(A(\Gamma), x) = \dim C(\Gamma) - 1$.

(iii) (Case 2) $A \cap V_0(\Gamma)$ is empty.

In this case, $D(A(\Gamma), x)$, is nothing to do with K or the tangent line of K. The original meaning of $D(A(\Gamma), x)$ is also the set of infinitesimal graphs concentrated at x. Precisely, $D(A(\Gamma), x)$ is the configuration space $C(A(\Gamma)) = \{(g, \Gamma') : g : A(\Gamma) \longrightarrow \Gamma'$ is an equivalence $\}$ quotiented by the translation and dilation relation defined in (case 1). Note: this relation reduces the dimension by 4, that is, $\dim D(A(\Gamma), x) = \dim C(A(\Gamma)) - 4$. Formally, $D(A(\Gamma), x)$ is independent of x, that is, $D(A(\Gamma), x) = D(A(\Gamma), x')$, for any x, x' in $\mathbb{R}^3$. But, similar to case 1, $\dim C(\Gamma; A) = \dim C(\Gamma/A) + \dim D(A(\Gamma), x)$, it is also equal to $\dim C(\Gamma) - 1$.

1.2.2. The canonical map $\Phi : C(\Gamma; A) \longrightarrow B_k$

Suppose Γ has k edge. Then Γ/A and $A(\Gamma)$ have the sum of edge numbers equal to k. Thus, for $\alpha = (g, g(\Gamma/A))$ in $C(\Gamma/A)$ and $\beta = (g', g'(A(\Gamma)))$ in $D(A(\Gamma), x)$, $\Phi(\alpha, \beta) = (\Phi(\alpha), \Phi(\beta))$, it is a well-defined element in B_k.

To show that $\Phi : C(\Gamma; A) \longrightarrow B_k$ is a continuous limit of $\Phi : C(\Gamma) \longrightarrow B_k$, let $\alpha + t\beta$ denote the following element in $C(\Gamma)$, as t is a small positive number:

Fix an element a_0 in A ($a_0 \in A \cap V_0(\Gamma)$, if $A \cap V_0(\Gamma) \neq \phi$).

$$g_t(v) = \begin{cases} g(v), & v \notin A, \\ g(a), & v = a_0, \\ g(a) + \frac{t}{|g'|}(g'(v) - g'(a_0)), & v \in A. \end{cases}$$

$$(|g'| = \max\{|g'(v) - g'(w)|,\ v, w, \in A\})$$

$\alpha + t\beta = (g_t, g_t(\Gamma))$, and it is easy to see that

$$\begin{aligned} &\lim_{t \to 0} (\alpha + t\beta) = (\alpha, \beta) \text{ and} \\ &\lim_{t \to 0} \Phi(\alpha + t\beta) = (\Phi(\alpha), \Phi(\beta)). \end{aligned}$$

In fact, $((g, g') \longrightarrow g_t)$ defines a collar of $C(\Gamma; A)$ in $C(\Gamma)$.

1.3. Identification Rules

We like to find all possible principle that which knot graphs have the same image in B_k, and construct the identification maps.

1.3.1 Free vertice

Suppose A has more than two vertices and $A(\Gamma)$ contains a free vertice v, that is, v is not an endpoint of any edge in $A(\Gamma)$.

Let $A_1 = A - \{v\}$ and we identify an element $(g, g(A(\Gamma)))$ in $D(A(\Gamma), x)$ with its restriction to A_1, $(g|_{A_1}, g(A_1(\Gamma)))$ in $D(A_1(\Gamma), x)$. We can also choose

an embedding ϕ from $D(A_1(\Gamma), x)$ into $D(A(\Gamma), x)$, which is a right inverse to the restriction, as follow:

$(g_1, g_1(A_1(\Gamma))) \in D(A_1(\Gamma), x)$,

let $||g_1|| = \max\{|g_1(v_1) - g_1(v_2)| : v_1, v_2 \in A_1\}$,

fix $v_0 \in A_1$, $g : A \longrightarrow \mathbb{R}^3$ defined by: $g(v) = g_1(v_0) + (2||g_1||, 0, 0)$, $g(v') = g_1(v')$, for $v' \neq v$, and $\phi(g_1, g_1(A_1(\Gamma)) = (g, g((A(\Gamma)))$. (Note: when $|A| = 2$, $||g_1|| = 0$.)

Let $\tau_0 : D(A(\Gamma), x) \longrightarrow D(A(\Gamma), x)$ be defined by $\tau_0(g, g(A(\Gamma))) = \phi(g|_{A_1}, g(A_1(\Gamma)))$. Thus, $D(A(\Gamma), x)$ is identified as its codimension 1 subset $\phi(A(A_1(\Gamma), x))$, and so is $C(\Gamma; A)$. This identification map $\tau_0 : C(\Gamma; A) \longrightarrow C(\Gamma; A)$ shall be called the identification map of type 0.

1.3.2 Univalent inner vertice

Suppose A has more than two vertices and $A(\Gamma)$ contains a univalent inner vertice v, that is, there exists a unique edge $e = \{v, w\}$ containing v as its end-point. Let $A_1 = A - \{v\}$, then $A_1(\Gamma)$ has one less edge than $A(\Gamma)$. We shall identify an element $(g, g(A(\Gamma))$ in $D(A(\Gamma), x)$ with $(g|_{A_1}, g(A_1(\Gamma)), \frac{g(v)-g(w)}{|g(v)-g(w)|})$ in $D(A_1(\Gamma), x) \times S^2$. Similar to the above case in 1.3.1, we may choose an embedding $\phi_1 : D(A_1(\Gamma), x) \times S^2 \longrightarrow D(A(\Gamma), x)$ such that $\phi_1(\alpha)$ is identified with α, for α in $D(A_1(\Gamma), x) \times S^2$. And the identification map $\tau_1 : D(A(\Gamma), x) \longrightarrow D(A(\Gamma), x)$ is defined as $\tau_1(g, g(A(\Gamma))) = \phi_1(g|_{A_1}, g(A_1(\Gamma)), \frac{g(v)-g(w)}{|g(v)-g(w)|})$. An explicit form of ϕ_1 is given in the following:

$\beta = (g_1, g_1(A_1(\Gamma))) \in D(A(\Gamma), x)$, $y \in S^2$,

let $||g_1|| = \max\{|g_1(v_1) - g_1(v_2)|$, v_1, v_2 in $A_1\}$,

$g : A \longrightarrow \mathbb{R}^3$, defined by $g(v) = g_1(w) + 2||g_1||y$, $g(v') = g_1(v')$, for $v' \neq v$.

Then, $\phi_1(\beta, y) = (g, g(A(\Gamma)))$, it is an element in $D(A(\Gamma), x)$.

Now, τ_1 is defined on the fibres $D(A(\Gamma), x)$ of $C(\Gamma; A)$, we extend τ_1 to the whole $C(\Gamma; A)$ in the obvious way. Thus, both D $(A$ (Γ), $x)$ and C $(\Gamma;$ $A)$

are identified by τ_1 into their proper subsets of codimension 1.

This identification map $\tau_1 : C(\Gamma; A) \longrightarrow C(\Gamma; A)$ shall be called the identification map of type I.

1.3.3 Bi-valent inner vertice

Suppose $A(\Gamma)$ has a bi-valent inner vertice v, that is, v is an inner vertice and there are exactly two edges e_1, e_2 which contain v as the endpoint.

Assume $e_1 = \{v, w_1\}$ and $e_2 = \{v, w_2\}$. The identification map on $D(A(\Gamma), x)$ (or $C(\Gamma; A)$) defined by:

$(g, g(A(\Gamma))) \in D(A(\Gamma), x)$,

let $\overline{g} : A \longrightarrow \mathbb{R}^3$ be: $\overline{g}(v) = g(w_1) + g(w_2) - g(v)$, and $\overline{g}(v') = g(v')$, for $v' \neq v$.

And, $\tau_2(g, g(A(\Gamma))) = (\overline{g}, \overline{g}(A(\Gamma)))$.

τ_2 is a B_k-map in the sense: $\Phi \circ \tau_2 = \Phi$, $\Phi : D(A(\Gamma), x) \longrightarrow B_k$ is the canonical B_k-structure. This identification map $\tau_2 : C(\Gamma; A) \longrightarrow C(\Gamma; A)$ shall be called the identification map of type II.

1.3.4 Edge Identification (Type III)

Suppose A is equal to an edge $e = \{v, w\}$ of Γ and A contains at least one inner vertice.

Then, the identification map $\tau_3 : D(A(\Gamma), x) \longrightarrow \mathbb{RP}^2$ is defined by : $(g, g(A(\Gamma))) \in D(A(\Gamma), x)$, $\tau_3 : (g, g(A(\Gamma))) = [g(v) - g(w)]$, and $C(\Gamma; A)$ is identified with $C(\Gamma/A) \times \mathbb{RP}^2$, by τ_3 also. This identification map $\tau_3 : C(\Gamma; A) \longrightarrow C(\Gamma/A) \times \mathbb{RP}^2$ is called the identification map of type III.

1.3.5 Non-edge Identification (Type IV)

Suppose A has exactly two vertices and A is not an edge in Γ.

Then, the identification map τ_4 is exactly the bundle projection from $C(\Gamma; A)$ onto $C(\Gamma/A)$. Note: $A(\Gamma)$ has no edge, this identification does not lose anything.

$\dim C(\Gamma; A) = \dim C(\Gamma/A)$, if and only if, A contains only base points.

When A contains at least one inner vertice, it could be thought as a "special case" of type 0, in spirit.

1.3.6 Summary of the identification maps

(i) $|A| \geq 3$, $\tau_0, \tau_1, \tau_2 : C(\Gamma; A) \longrightarrow C(\Gamma; A)$.

(ii) $|A| = 2$, $\tau_3, \tau_4 : C(\Gamma; A) \longrightarrow C(\Gamma/A) \times P(A)$,

where

$$P(A) = \begin{cases} \mathbb{RP}^2, & \text{if } A(\Gamma) \text{ has one edge,} \\ \{\text{one point}\}, & \text{if } A(\Gamma) \text{ has no edge.} \end{cases}$$

(iii) τ_0, τ_1 send $C(\Gamma; A)$ into codimension 1 subset; $\tau_4(C(\Gamma; A))$ has dimension $\leq \dim C(\Gamma) - 2$, if $A \not\subset V_0(\Gamma)$.

1.4 Stable configuration space

For any positive integer q, let $C(\Gamma)^{(q)} = C\ (\Gamma) \times B_q$, where $B_q = \prod_q \mathbb{RP}^2/\Sigma_q$. An element in $C(\Gamma)^{(q)}$ is a knot graph $(g,\ \Gamma')$ together with q points $\xi_1,\ \xi_2,\ \cdots,\ \xi_q$ in $\mathbb{RP}^2$. $C(\Gamma)^{(q)}$ is called the q-stable extension of $C(\Gamma)$. Similarly, $C(\Gamma; A)^{(q)} = C(\Gamma; A) \times B_q$.

1.4.1 Identifications

Following section 1.3.6, we have the identification maps directly:

(i) $|A| \geq 3$, $\tau_0, \tau_1, \tau_2 : C(\Gamma; A)^{(q)} \longrightarrow C(\Gamma; A)^q$

(ii) $|A| = 2$, $\tau_3, \tau_4 : C(\Gamma; A)^{(q)} \longrightarrow C(\Gamma/A)^{(q+e(A))}$,

where $e(A)$ is the number of edges in $A(\Gamma)$.

1.4.2 Combined configuration spaces

Definition (1.13): Suppose Γ has no free vertice. order $(\Gamma) = |E(\Gamma)| - |V_1(\Gamma)|$, that is, the number of edges in Γ minus the number of inner vertices in Γ.

Definition (1.14): Suppose Γ is a knot graph with order n and k edges. $S(\Gamma)$ is the $(3n-k)$-stable extension of $C(\Gamma)$, i.e. $C(\Gamma)^{(3n-k)}$. (it is a B_{3n}-space).

(1.15): Let $\overline{\mathcal{S}}(n)$ be disjoint union of $S(\Gamma)$, for all Γ with order n, and $\mathcal{S}(n)$ be the quotient space of $\overline{\mathcal{S}}(n)$ by taking all possible identifications of type 0, I, II, III and IV. Thus $\mathcal{S}(n)$ has a canonical map

$$\Phi : \mathcal{S}(n) \longrightarrow B_{3n}.$$

Is there any cycle α in $\mathcal{S}(n)$ such that $\Phi_*(\alpha) \neq 0$ in $H_*(B_{3n})$?

§2 $P(k)$-structure

2.1 The canonical $P(k)$-structures

$P(k)$ denotes the space $\prod\limits_{k} \mathbb{RP}^2$, the product of k copies of $\mathbb{RP}^2$.

Assume Γ has k edges. Choose an order for the edges $(e_1, e_2, \cdots, e_k)$, $e_i = \{v_i, w_i\}$, $i = 1, 2, \cdots, k$. Let $\Psi : C(\Gamma) \longrightarrow P(k)$ be the continuous map defined by: for $(g, g(\Gamma)) \in C(\Gamma)$, $g(\Gamma)$ is knot graph on K,

$$\Psi(g, g(\Gamma)) = ([g(v_1) - g(w_1)], [g(v_2) - g(w_2)], \cdots, [g(v_k) - g(w_k)]).$$

Changing the order of edges, we have $k!$ different maps from $C(\Gamma)$ to $P(k)$. Let $\Sigma_k \cdot \Psi$ denote the set of $k!$ maps, it is the orbit of Σ_k-action. All the maps in $\Sigma_k \cdot \Psi$ are called the canonical $P(k)$-structures of $C(\Gamma)$.

Now, consider the boundary $C(\Gamma; A)$, it is a fibre bundle over $C(\Gamma/A)$ with fibre $D(A(\Gamma), x)$. Assume Γ/A has k_1 edges and $A(\Gamma)$ has k_2 edges, $k_1 + k_2 = k$.

$\Psi_1 : C(\Gamma/A) \longrightarrow P(k_1)$ denotes a canonical $P(k_1)$-structure of $C(\Gamma/A)$, and $\Psi_2 : D(A(\Gamma), x) \longrightarrow P(k_2)$ denotes a canonical $P(k_2)$-structure of $D(A(\Gamma), x)$, then the restriction of Ψ to $C(\Gamma; A)$, $\Psi|_{C(\Gamma;A)} = \sigma \cdot (\Psi_1, \Psi_2)$, for some permutation σ in Σ_k.

Note: To make sure the equality, $k_1 + k_2 = k$, the ordered edge set $(e_1, e_2, \cdots, e_{k_1})$ of $E(\Gamma/A)$ may have the same edge in different coordinates, that is, one edge may appear more one times in the ordered edge set. And, when the knot graph has multi-edge, $\Sigma_k \cdot \Psi$ does not have $k!$ different maps. But, there is no essential influence on the theories of this article.

2.2 Stable $P(r)$-extension of configuration space

Definition (2.1): Suppose Γ is a knot graph with k edges and r is a positive integer. $C(\Gamma) \times P(r)$ is called the $P(r)$-extension of $C(\Gamma)$. If A is a subset of $V(\Gamma)$, $D(A(\Gamma), x) \times P(r)$ is called the $P(r)$-extension of $D(A(\Gamma), x)$.

We are interested in the $P(r)$-extension of $C(\Gamma)$ and their canonical $P(k+r)$-structure $\Sigma_{k+r} \cdot \Psi$, where $\Psi : C(\Gamma) \times P(r) \longrightarrow P(k+r)$ is defined by:

$$(g, g(\Gamma)) \in C(\Gamma), (\xi_1, \xi_2, \cdots, \xi_r) \in P(r),$$
$$\Psi((g, g(\Gamma)), \xi_1, \xi_2, \cdots, \xi_r) = (\Psi(g, g(\Gamma)), \xi_1, \xi_2, \cdots, \xi_r).$$

Similarly, for $D(A(\Gamma), x) \times P(r)$.

Now, consider the $P(r)$-extension of $C(\Gamma)$ prescribed a canonical $P(k+r)$-structure, and the disjoint union of the all possible spaces is denoted by $C(\Gamma) \times P(r) \times \Sigma_{k+r} \cdot \Psi$, where $\Sigma_{k+r} \cdot \Psi$ is a discrete space.

Definition (2.2): Suppose Γ is a knot graph on K, the order of Γ is n and Γ does not have any free vertice.

$$P(\Gamma) = C(\Gamma) \times P(3n - k) \times \Sigma_{3n} \cdot \Psi,$$

where $k = |E(\Gamma)|$ and Ψ is a canonical $P(3n)$-structure of $C(\Gamma) \times P(3n - k)$.

In the following, we consider the identifications between the spaces $P(\Gamma)$, or more general, $C(\Gamma) \times P(r) \times \Sigma_{k+r} \cdot \Psi$. The identification maps in section 1.3 have two forms:

(i) $|A| \geq 3$, $\tau : C(\Gamma; A) \longrightarrow C(\Gamma; A)$
(type 0, type I or type II)

(ii) $|A| = 2$, $\tau : C(\Gamma; A) \longrightarrow C(\Gamma/A) \times P(A)$
(type III or type IV)
$(P(A) = P(|E(A(\Gamma))|).)$

All the identification maps directly give identification maps for $C(\Gamma) \times P(r)$, that is,

$$\tau \times id : \begin{cases} C(\Gamma; A) \times P(r) \longrightarrow C(\Gamma; A) \times P(r), \\ C(\Gamma; A) \times P(r) \longrightarrow C(\Gamma/A) \times P(A) \times P(r). \end{cases}$$

Let $\overline{\tau} = \tau \times id_{P(r)}$, for each case.

Theorem (2.3): Suppose Γ is a knot graph and a subset A of $V(\Gamma)$ satisfies the condition of type 0, I, II, III, or IV in section 1.3. $\overline{\tau}$ is the corresponding identification map above. Ψ is a canonical $P(k+r)$-structure for $C(\Gamma; A) \times P(r)$ and Ψ' is a canonical $P(k+r)$-structure for $\overline{\tau}(C(\Gamma; A) \times P(r))$. Then, there is a $1-1$-correspondence

$$\lambda : \Sigma_{k+r} \cdot \Psi \longrightarrow \Sigma_{k+r} \cdot \Psi'$$

such that, for any Ψ_1 in $\Sigma_{k+r} \cdot \Psi$, $\lambda(\Psi_1) \circ \overline{\tau} = \Psi_1$ on $C(\Gamma; A) \times P(r)$. In fact, λ satisfies $\lambda(\sigma \cdot \Psi_1) = \sigma \cdot \lambda(\Psi_1)$, for any $\sigma \in \Sigma_{k+r}$ and $\Psi_1 \in \Sigma_{k+r} \cdot \Psi$, that is, λ is Σ-equivariant.

(2.4): Identification Rule

$$\begin{aligned} &\widetilde{\tau} : C(\Gamma; A) \times P(r) \times \Sigma_{k+r} \cdot \Psi \longrightarrow C(\Gamma; A) \times P(r) \times \Sigma_{k+r} \cdot \Psi' \\ &\alpha \in C(\Gamma; A) \times P(r), \Psi_1 \in \Sigma_{k+r} \cdot \Psi, \\ &\widetilde{\tau}(\alpha, \Psi_1) = (\overline{\tau}(\alpha), \lambda(\Psi_1)). \end{aligned}$$

(Similar for the other case)

Definition (2.5): A knot graph Γ is said to be normal if Γ does not have any one of the following three kinds of vertices: (i) free vertice, (ii) univalent inner vertice, (iii) bi-valent inner vertice.

(2.6): Suppose n is a positive integer. Let $\overline{\mathcal{P}(n)}$ be the disjoint union of $P(\Gamma)$, for all normal knot graphs Γ with order n, and $\mathcal{P}(n)$ be the space $\overline{\mathcal{P}(n)}$ quotiented by all possible identifications of type 0, I, II, III and IV (given by the maps $\widetilde{\tau}$ on the boundary of $P(\Gamma)$'s).

Therefore, $\mathcal{P}(n)$ is a $P(3n)$-space, and has a canonical $P(3n)$-structure $\widetilde{\Psi} : \mathcal{P}(n) \longrightarrow P(3n)$, defined by: for $\alpha \in C(\Gamma) \times P(3n-k)$ and Ψ_1, a canonical $P(3n)$-structure for $C(\Gamma) \times P(3n-k)$,

$$\widetilde{\Psi}(\alpha, \Psi_1) = \Psi_1(\alpha).$$

§3 Infinitesimal knot graph

Let l_0 denote the line of z-axis, $\{(0,0,t) : t$ is a real number, and consider the configuration space $C(\Gamma, l_0)$ of knot graphs on l_0. We shall define the translation and dilation on all $C(\Gamma, l_0)$ and the extended translation and dilation relation on part of $C(\Gamma, l_0)$, for which Γ satisfies a splittable condition.

3.1 The translation and dilation relation

Definition (3.1): Assume $(g, g(\Gamma))$ and $(g', g'(\Gamma))$ are two elements in $C(\Gamma, l_0)$. $(g, g(\Gamma))$ and $(g', g'(\Gamma))$ are equivalent under the translation and dilation relation, if there are real number λ and t, $\lambda > 0$, such that $g(v) = \lambda g'(v) + (0,0,t)$, for all $v \in V(\Gamma)$. We may write the relation as $(g, g(\Gamma)) \overset{\text{t.d.}}{\sim} (g', g'(\Gamma))$.

Notation (3.2): $W_0(\Gamma)$ denotes the quotient space of $C(\Gamma, l_0)$ by the translation and dilation relation. Thus,

$$\dim W_0(\Gamma) = \dim C(\Gamma, l_0) - 2$$

Definition (3.3): A knot graph Γ is said to be splittable, if there are subgraph Γ_1 and Γ_2 in Γ such that the following two conditions hold:

(i) $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$.

(ii) $\Gamma_1 \cap \Gamma_2$ is either empty or a knot graph consisting of exactly one base point (no edge and no inner point).

And Γ is said to be splited into Γ_1 and Γ_2

Now, suppose Γ is splited into Γ_1 and Γ_2. Let $\rho : W_0(\Gamma) \longrightarrow W_0(\Gamma_1) \times W_0(\Gamma_2)$ be the map defined by $\rho(g, g(\Gamma)) = ((g|_{\Gamma_1}, g(\Gamma_1)), (g|_{\Gamma_2}, g(\Gamma_2)))$.

Definition (3.4): Suppose Γ is splited into Γ_1 and Γ_2. Two elements α and β in $W_0(\Gamma)$ is said to be equivalent under the extended translation and dilation relation, (simply, ETD relation), if $\rho(\alpha) = \rho(\beta)$ in $W_0(\Gamma_1) \times W_0(\Gamma_2)$.

Note: If $\Gamma_1 \cap \Gamma_2$ is empty, $\dim(W_0(\Gamma_1) \times W_0(\Gamma_2)) = \dim W_0(\Gamma) - 2$. If $\Gamma_1 \cap \Gamma_2$ consists of a base point, $\dim(W_0(\Gamma_1) \times W_0(\Gamma_2)) = \dim W_0(\Gamma) - 1$. Thus, $W_0(\Gamma)/ETD$ has dimension at most $\dim C(\Gamma) - 3$, and is also a $P(k)$-space.

Now, we consider the boundary of $W_0(\Gamma)$. For any A, a proper subset of $V(\Gamma)$, that is, $A \subset V(\Gamma)$ and $A \neq V(\Gamma)$, let $W_0(\Gamma; A)$ denote the quotient space of $C(\Gamma; A)$ (for the knot graphs on l_0) by the T.D. relation. Because l_0 is a straight line, $C(\Gamma; A) = C(\Gamma/A) \times D(A(\Gamma), x)$ and $D(A(\Gamma), x)$ is exactly $W_0(A(\Gamma))$. Thus, $W_0(\Gamma; A) = W_0(\Gamma/A) \times W_0(A(\Gamma))$, which has dimension equal to $\dim W_0(\Gamma) - 1$.

When $A(\Gamma)$ has no base point, the T.D. relation for $C(A(\Gamma))$ is modified as follows:

$(g, g(A(\Gamma))) \overset{\text{t.d.}}{\sim} (g', g'(A(\Gamma)))$, if there exist real number λ and $y \in \mathbb{R}^3$ such that $g(v) = \lambda g'(v) + y$, for all $v \in A$.

$$\begin{aligned} W_0(A(\Gamma)) \qquad &= C(A(\Gamma), l_0)/\text{T.D. relation} \\ &= D(A(\Gamma), x), \text{for any } x. \end{aligned}$$

Thus,

$$\dim W_0(A(\Gamma)) = \dim C(A(\Gamma)) - 4.$$

The extend T.D. relation could happen in $W_0(\Gamma/A)$ and in $W_0(A(\Gamma))$. When $A(\Gamma)$ has no base point, $A(\Gamma)$ is splittable, if and only if, $A(\Gamma)$ is a disjoint union of $A_1(\Gamma)$ and $A_2(\Gamma)$, for some A_1 and A_2. For example, Γ is the following knot graph: $V_0(\Gamma) = \{ x_i,\ i = 1,2,3,4,5,6,7 \}$, $V_1(\Gamma) = \{ y_j,\ j = 1,2,3,4,5 \}$ and $E(\Gamma) = \{ \{x_1, y_1\}, \{x_2, y_1\}, \{x_3, y_2\}, \{x_4, y_3\}, \{x_5, y_4\}, \{x_6, y_5\}, \{x_7, y_5\}, \{y_1, y_2\}, \{y_2, y_3\}, \{y_3, y_4\}, \{y_4, y_5\} \}$. $A = \{y_1, y_2, y_4, y_5\}$, $A_1 = \{y_1, y_2\}$, $A_2 = \{y_4, y_5\}$. Then $A(\Gamma)$ is the splited into $A_1(\Gamma)$ and $A_2(\Gamma)$.

(3.5): If $A(\Gamma)$ is the disjoint union of $A_1(\Gamma), A_2(\Gamma), \cdots, A_r(\Gamma)$, then the extended translation and dilation relation associated with the splitting

$(A_1(\Gamma), A_2(\Gamma), \cdots, A_r(\Gamma)$ is defined as following: $\alpha = (g, g(A\Gamma)))$, $\alpha' = (g', g'(A(\Gamma)))$ are two element in $W_0(A(\Gamma))$,

$$\alpha \overset{\text{E.T.D.}}{\sim} \alpha', \text{ if } \ g|_{A_i(\Gamma)} \overset{\text{T.D.}}{\sim} g'|_{A_i(\Gamma)},$$

3.2 The combined space $\mathcal{W}_0(n)$

Similar to the construction of $\mathcal{P}(n)$, let $\overline{\mathcal{P}}(n, l_0)$ be the disjoint union of $P(\Gamma)$ of l_0, for all normal knot graph Γ with order n, and $\mathcal{P}(n, l_0)$ be the quotient space of $\overline{\mathcal{P}}(n, l_0)$ by the identifications of type 0, I, II, III or IV, defined in section 1.3.

And, $\mathcal{W}_0(n)$ be the quotient space of $\mathcal{P}(n, l_0)$ by the T.D. relations and the possible E.T.D. relations.

We can understand $\mathcal{W}_0(n)$ as the quotient space of disjoint union of $W_0(\Gamma) \times P(3n-k) \times \Sigma_{3n} \cdot \Psi$, with further quotients by the E.T.D. relations.

Note: The E.T.D. relations considered include all the E.T.D. relation for configuration spaces and for the boundaries ($W_0(\Gamma/A)$ or $W_0(A(\Gamma))$.

3.3 The S^1-action on $\mathcal{W}_0(n)$

Identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$, S^1 may acts on $\mathbb{R}^3$ by the multiplication on the complex plane part, that is, the rotation about the z-axis.

Thus, S^1 also acts on $C(\Gamma, l_0)$: $\lambda \in S^1$, $(g, g(\Gamma)) \in C(\Gamma, l_0)$, $\lambda \cdot (g, g(\Gamma)) = (\lambda g, (\lambda g)(\Gamma))$.

It is easy to see that all the identification maps of any type in section 1.3 are S^1-equivariant maps (for the case of l_0) and T.D. and E.T.D. relations are also compatible with the S^1-action. Thus S^1 also acts on $\mathcal{W}_0(n)$. Furthermore, the S^1-action on $\mathbb{R}^3$ also induces an S^1-action on $\mathbb{RP}^2$ and $P(k)$,

and hence, the $P(k)$-structure map Ψ of $\mathcal{W}_0(n)$ is S^1-equivariant.

Proposition (3.6):

(i) $W_0(\Gamma)$, $\mathcal{W}_0(n)$ are semifree S^1-space.

(ii) $P(k)$ has a unique fixed point of the S^1-action.

(Proof?)

Notation (3.7): $\mathcal{H}(n)$ denotes the set of all fixed points of S^1-action on $\mathcal{W}_0(n)$, and $\mathcal{D}(n)$ denotes the normal vector bundle of $\mathcal{H}(n)$ in $\mathcal{W}_0(n)$.

We find the vector bundle $\mathcal{D}(n)$ over $\mathcal{H}(n)$ has finite structure group. Roughly speaking, the reason is the following: Let $H(\Gamma)$ be the fixed point set of $W_0(\Gamma)$. It is not hard to find that $\mathcal{H}(n)$ is the union of $H(\Gamma)$ and the restriction of the vector bundle $\mathcal{D}(n)$ to $H(\Gamma)$ is trivial. And the identifications of type 0 to IV make the structure group isomorphic the semi-direct product of $\bigoplus_r \mathbf{Z_2}$ and Σ_r, for some r.

If we consider the S^1-equivariant map $\Psi : \mathcal{W}_0(n) \longrightarrow P(3n)$. ξ_0 denotes the unique fixed point in $P(3n)$. Then $\mathcal{H}(n) = \Psi^{-1}(\xi_0)$ and $\mathcal{D}(n)$ is a complex vector bundle of $2n$-dim, that is, fibre $= \mathbb{C}^{2n}$. $\dim \mathcal{H}(n) = \dim H(\Gamma) = 2n-2$, for non-splittable knot graph Γ.

3.4 The space of infinitesimal knot graphs

For any $x \in S^2$, l_x denotes the line $\{tx, t \text{ is a real number }\}$ with linear order : $t_1 x < t_2 x$, for $t_1 < t_2$. Let $W_x(\Gamma) = C(\Gamma, l_x)/\text{T.D.relation}$ and $W(\Gamma) = \bigcup_{x \in S^2} W_x(\Gamma)$. Then $W(\Gamma)$ is a fibre bundle over S^2 with fibre $W_0(\Gamma)$.

Consider the following $SO(3)$-action on $W(\Gamma)$. $\sigma \in SO(3)$, $(g, g(\Gamma)) \in W_x(\Gamma)$,

$$\sigma \cdot (g, g(\Gamma)) = (\sigma \cdot g, (\sigma \cdot g)(\Gamma)) \in W_{\sigma(x)}(\Gamma),$$

where $(\sigma \cdot g)(v) = \sigma(g(v))$, for $v \in V(\Gamma)$.

Notation (3.8): For any S^1-space X, let $SO(3) \underset{S^1}{\times} X$ denote the quotient space $SO(3) \times X/(\sigma, x) \sim (\sigma\lambda^{-1}, \lambda x)$, for $\sigma \in SO(3)$, $x \in X$, and $\lambda \in S^1$, where S^1 is identified with the $SO(2)$-subgroup of $SO(3)$, which rotates about the z-axis.

Thus, $SO(3) \underset{S^1}{\times} W_0(\Gamma)$ is naturally diffeomorphic to $W(\Gamma)$ by the map sending $(\sigma, \alpha) \longrightarrow \sigma \cdot \alpha$.

$SO(3)$ also acts on $P(k)$. The map of $P(k)$-structure of $W_0(\Gamma)$, $\Psi : W_0(\Gamma) \longrightarrow P(k)$, is a S^1-equivariant map. Ψ induces a $SO(3)$-equivariant map $\widetilde{\Psi} : SO(3) \underset{S^1}{\times} W_0(\Gamma) \longrightarrow P(k)$, $\widetilde{\Psi}(\sigma, \alpha) = \sigma \cdot \Psi(\alpha)$.

$\widetilde{\Psi}$ is exactly the canonical $P(k)$-structure of $W(\Gamma)$. Thus, we conclude the following:

(3.9): $W_0(\Gamma)$ and its S^1-equivariant $P(k)$-structure map $\Psi : W_0(\Gamma) \longrightarrow P(k)$ completely determines the $P(k)$-space $W(\Gamma)$.

With some abuse of notation, we may use $\Sigma_k \cdot \Psi$ to represent the set of all canonical $P(k)$-structure of $W(\Gamma)$ and let $\mathcal{W}(n)$ denote the quotient space of disjoint union of $W(\Gamma) \times P(3n-k) \times \Sigma_{3n} \cdot \Psi$. Thus, $\mathcal{W}(n)$ is also equal to $SO(3) \underset{S^1}{\times} \mathcal{W}_0(n)$. And, the $P(3n)$-structure of $\mathcal{W}(n)$ is also completely determined by the S^1-equivariant $P(3n)$-structure map $\Psi : \mathcal{W}_0(n) \longrightarrow P(3n)$, as above. This suggests that the study of $\mathcal{W}_0(n)$ and the associated S^1-equivariant $P(3n)$-structure map will completely understand the "degree" of $\mathcal{W}(n)$, in the dimension $6n$.

The following Theorem will be proved in the next two chapter.

Theorem (3.10): The degree homomorphism

$$\widetilde{\Psi}_* : H_{6n}(\mathcal{W}(n), \widetilde{\Psi}^{\#}(\mathcal{O}_{3n})) \longrightarrow H_{6n}(P(3n); \mathcal{O}_{3n}) \approx \mathbf{Z}$$

"become" a characteristic class of the complex vector bundle $\mathcal{D}(n)$ over $\mathcal{H}(n)$. (Thus, if $\mathcal{D}(n)$ has finite structure group, then the degree homomorphism is

zero.)

In the next chapter, we shall develop a theory for the $P(k)$-space with S^1-equivariant $P(k)$-structure.

Remark: In $P(3n)$, let us consider the open set $\{(y_1, y_2, \cdots, y_{3n}) \in P(3n) : y_i$ is not perpendicular to the z-axis, for all $i = 1, 2, \cdots, 3n\}$, it is the canonical $\mathbb{R}^{6n}$ neighborhood of the fixed point ξ_0. Then the normal bundle $\mathcal{D}(n)$ can be identified with $\Psi^{-1}(\mathbb{R}^{6n})$ canonically, and $\Psi : \mathcal{D}(n) \longrightarrow \mathbb{R}^{6n}$ is an injective linear map on each fibre of $\mathcal{D}(n)$.

§4 S^1-equivariant $P(k)$-structure

Suppose (X, f) is a $P(k)$-space, X has an S^1-action and $f : X \longrightarrow P(k)$ is an S^1-equivariant map. Let $SO(X) = SO(3) \underset{S^1}{\times} X$, then $SO(X)$ is also a $P(k)$-space. In this part, we study the relation between the degree homomorphism of $P(k)$-space $(SO(X), \widetilde{f})$ and the homomorphism $\overline{f}_* : H_*(X/S^1) \longrightarrow H_*(P(k)/S^1)$, and furthermore.

4.1 The S^1-action on $P(k)$

$P(k) = \prod\limits_k \mathbb{RP}^2$. For any $\lambda = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, and $(x, y, z) \in \mathbb{R}^3$, $\lambda \cdot (x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$. This gives S^1-action on S^2 and $\mathbb{RP}^2$. For $\prod\limits_k S^2$ and $\prod\limits_k \mathbb{RP}^2$, S^1 acts on them diagonally, that is, $\lambda \cdot (y_1, y_2, \cdots, y_k) = (\lambda \cdot y_1, \lambda \cdot y_2, \cdots, \lambda \cdot y_k)$. Suppose X is a S^1-space. x_0 in X is said to the fixed point of the S^1-action, if $\lambda \cdot x_0 = x_0$, for all $\lambda \in S^1$. The S^1-action on X is said to be free, if $\lambda \cdot x \neq x$, for all $\lambda \neq 1$, and for all $x \in X$. The S^1-action on X is said to be semifree, if X-(the fixed point set of the S^1-action) is free. The following are the main S^1-spaces we need:

(i) The S^1-action on $\prod\limits_k S^2$ is semifree, and its fixed points set is $\prod\limits_k$ (fixed point set of S^2), which has 2^k elements.

(ii) The S^1-action on $P(k) = \prod\limits_k \mathbb{RP}^2$ has only one fixed point.

Defintion (4.1): For any S^1-space X, let $SO(X)$ denote the quotient space $SO(3) \underset{S^1}{\times} X$, that is, $SO(3) \times X$ quotiented by the equivariant relation: for $\sigma \in SO(3)$, $x \in X$, $\lambda \in S^1$, $(\sigma, \lambda x) \sim (\sigma\lambda^{-1} \cdot x)$.

(S^1 is the identified as the subgroup of rotations about the z-axis in $SO(3)$.)

Let Fix (X) denote the set of fixed points in the S^1-space X. For any $i > \dim \mathrm{Fix}\,(X)$, there is a homomorphism β_1 from $H_i(X/S^1)$ to $H_{i+1}(X)$.

Similarly, $SO(3) \times X$ is a S^1-space : $\lambda \cdot (\sigma, x) = (\sigma\lambda^{-1}, \lambda \cdot x)$, there is a homomorphism $\beta_2 : H_{i+3}(SO(X)) \longrightarrow H_{i+4}(SO(3) \times X)$ and $\overline{\beta} : H_{i+1}(X) \longrightarrow H_{i+4}(SO(3) \times X)$.

Consider the following diagram

$$\begin{array}{ccc} H_{i+1}(X) & \xrightarrow{\overline{\beta}} & H_{i+4}(SO(3) \times X) \\ \uparrow{\scriptstyle \beta_1} & & \uparrow{\scriptstyle \beta_2} \\ H_i(X/S^1) & \xrightarrow{\beta} & H_{i+3}(SO(X)) \end{array}$$

It is obvious that $Im(\overline{\beta} \circ \beta_1) \subset Im(\beta_2)$.

Question: Is there a homomorphism $\beta : H_i(X/S^1) \longrightarrow H_{i+3}(SO(X))$ such that the above diagram is commutative? Actually, β is completely an analogue of β_1. $SO(X)$ is an $SO(3)$-space, $\sigma[\sigma_1, x)] = [(\sigma\sigma_1, x)]$, and orbit space $SO(X)/SO(3)$ is exactly X/S^1. If we assume that the S^1-action on X is semifree, then the fibre map $SO(X) \longrightarrow X/S^1$ has fibres homeomorphism to $SO(3)$ except over the fixed points. Therefore, the homomorphism $\beta : H_i(X/S^1) \longrightarrow H_{i+3}(SO(X))$ exists and satisfies the functorial property: for any S^1-equivariant map $f : X \longrightarrow Y$ of semifree S^1-spaces X and Y, $i > \dim(\mathrm{Fix}\,(X))$ (and $\dim(\mathrm{Fix}\,(Y))$, the following diagram commutes:

$$\begin{array}{ccc} H_i(X/S^1) & \longrightarrow & H_{i+3}(SO(X)) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} \\ H_i(Y/S^1) & \longrightarrow & H_{i+3}(SO(Y)) \end{array}$$

Why we need the condition $i > \dim \mathrm{Fix}\,(X)$?

When $i > \dim \mathrm{Fix}\,(X)$, we may assume there is a cell decomposition of X/S^1, such that $\mathrm{Fix}\,(X) \subset$ the $(i-1)$-skeleton of X/S^1. In a corresponding

cell decomposition in X, the behavior of Fix(X) will not affect the boundary operator of $(i+1)$-cells.

Remark: If $\dim X = i+1$, then β_1,β_2 and $\overline{\beta}$ are all isomorphism and the existence of β is obvious. When applying the theory in Chapter 5, it is the situation.

4.2 $P(k)$-space with a semifree S^1-action

Suppose X is a semifree S^1-space and $f : X \longrightarrow P(k)$ is an S^1-equivariant map. ξ_0 denotes the unique fixed point of S^1-action on $P(k)$ and Y denotes the inverse image $f^{-1}(\xi_0)$. Furthermore, let U_0 be an S^1-invariant contractible open neighborhood of ξ_0 in $P(k)$ and $U_1 = f^{-1}(U_0)$, it is an S^1-invariant open neighborhood of Y in X. Then, $U_0 - \xi_0$ and $U_1 - Y$ are free S^1-spaces.

Let $\overline{f} : X/S^1 \longrightarrow P(k)/S^1$ be the map induced by f, and $\widetilde{f} : SO(X) \longrightarrow P(k)$ be the map defined by:

$$\widetilde{f}(\sigma, x) = \sigma \cdot f(x).$$

Let $\mathcal{O}$ be the orientation sheaf of $P(k)$, $\overline{\mathcal{O}}$ be the orientation sheaf of $P(k)/S^1$ such that $\pi^{\#}(\overline{\mathcal{O}}) = \mathcal{O}$, where $\pi : P(k) \longrightarrow P(k)/S^1$ is the quotient map.

The homomorphism β described in the last section 4.1, together with the local coefficient $\mathcal{O}$ give the following diagram:

$$
\begin{tikzcd}
H_{2k-3}(X/S^1, \overline{f}^{\#}(\overline{\mathcal{O}})) \arrow[r, "\beta"] \arrow[d, "\overline{f}_*"] & H_{2k}(SO(X), \widetilde{f}^{\#}(\mathcal{O})) \arrow[dd, "\widetilde{f}_*"] \arrow[dr, "SO(f)_*"] & \\
H_{2k-3}(P(k)/S^1, \overline{\mathcal{O}}) \arrow[rr, "\beta"] \arrow[dr, "\overline{\beta}"'] & & H_{2k}(SO(P(k)), \mathcal{O}') \arrow[dl, "\gamma"] \\
& H_{2k}(P(k), \mathcal{O}) &
\end{tikzcd}
$$

When $\overline{f}$ restricts to $U_1 - Y/S^1$, we have $\overline{\overline{f}}\colon (U_1 - Y/S^1) \longrightarrow (U_0 - \xi_0/S^1)$.

Consider

$$
\begin{aligned}
\overline{\overline{f}}_*\colon H_{2k-4}(U_1 - Y/S^1) \longrightarrow \quad & H_{2k-4}(U_0 - \xi_0/S^1) \\
\approx \quad & H_{2k-4}(\mathbb{C}\mathbb{P}^{k-1}) \\
\approx \quad & \mathbf{Z}.
\end{aligned}
$$

Theorem (4.2): If $\overline{\overline{f}}_*\colon H_{2k-4}(U_1 - Y/S^1) \longrightarrow H_{2k-4}(U_0 - \xi_0/S^1)$ is a zero-homomorphism, then the degree homomorphism $\widetilde{f}_* : H_{2k}(SO(X), \widetilde{f}^{\#}(\mathcal{O})) \longrightarrow H_{2k}(P(3n), \mathcal{O})$ sends $\beta(x)$ to 0, for all $x \in H_{2k-3}(X/S^1, \overline{f}^{\#}(\overline{\mathcal{O}}))$. (That is, $\overline{\overline{f}}_* = 0 \Longrightarrow \widetilde{f} \circ \beta = 0$.)

Note: In the statement, we do not need the commutative property: $SO(f)_* \circ \beta = \overline{f}_* \circ \beta_*$, probably, it is true, but I never prove it. (because $P(k)$ is not semifree)

4.3 Preliminary to the proof of Theorem (4.2)

$\prod_k S^2$ is the universal covering space of $P(k)$, let $S(k) = \prod_k S^2$ and $q : S(k) \longrightarrow P(k)$ be covering projection.

Let M the pullback of $S(k)$ by $f : X \longrightarrow P(k)$, that is, $M = \{(x, y) \in X \times S(k) : f(x) = q(y)\}$. Thus, we have the following commutative diagram.

(4.3)
$$\begin{array}{ccc} M & \xrightarrow{g} & S(k) \\ \Big\downarrow{\scriptstyle q'} & & \Big\downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & P(k) \end{array}$$

$S(k)$ is also a S^1-space and q is a S^1-equivariant map. S^1 acts on $X \times S(k)$ by: $\lambda \in S^1$, $\lambda \cdot (x, y) = (\lambda x, \lambda y)$.

If $(x, y) \in M$, $\lambda \cdot (x, y)$ is also in M. This defines an S^1-action on M such that both $g : M \longrightarrow S(k)$ and the covering projection $q' : M \longrightarrow X$ are S^1-equivariant. This means that the diagram (4.3) is a commutative diagram of S^1-equivariant maps. This induces the following commutative diagram of $SO(3)$-equivariant diagram.

(4.4)
$$\begin{array}{ccc} SO(M) & \xrightarrow{\widetilde{g}} & S(k) \\ \Big\downarrow{\scriptstyle SO(q')} & & \Big\downarrow{\scriptstyle q} \\ SO(X) & \xrightarrow{\widetilde{f}} & P(k) \end{array}$$

(Note: q is originally an $SO(3)$-equivariant map).

Because $q' : M \longrightarrow X$ is an S^1-equivariant covering projection and X is assumed to be a semifree S^1-space, M is also a semifree S^1-space. Thus, both X and M provide β-homomorphism from $H_{2k-3}((\cdot)/S^1)$ to $H_{2k}(SO(\cdot))$. They form the following commutative diagram

$$
\begin{array}{ccccc}
H_{2k-3}(M/S^1) & \xrightarrow{\beta_M} & H_{2k}(SO(M)) & \xrightarrow{\widetilde{g}_*} & H_{2k}(S(k)) \\
\downarrow{\overline{q'}_*} & & \downarrow{SO(q')_*} & & \downarrow{q_*} \\
H_{2k-3}(X/S^1, \overline{f}^{\#}(\overline{\mathcal{O}})) & \xrightarrow{\beta_X} & H_{2k}(SO(X), \widetilde{f}^{\#}(\mathcal{O})) & \xrightarrow{\widetilde{f}_*} & H_{2k}(P(k), \mathcal{O})
\end{array}
\tag{4.5}
$$

(The local coefficients for M/S^1, $SO(M)$, and $S(k)$ are trivial.)

Proposition (4.6): $\widetilde{f}_* \circ \beta_X = 0$, if and only if, $\widetilde{g}_* \circ \beta_M = 0$.

Proof: Because $H_{2k}(S(k)) \approx H_{2k}(P(k), \mathcal{O}) \approx \mathbf{Z}$, it is enough to show this proposition for the coefficient over rationals Q. But, when over rational, both $q_* : H_{2k}(S(k)) \longrightarrow H_{2k}(P(k), \mathcal{O})$, and $\overline{q'}_* : H_{2k-3}(M/S^1) \longrightarrow H_{2k-3}(X/S^1)$ are isomorphisms. By the commutative property of (4.5), we prove this proposition. (Note: $M/S^1 \longrightarrow X/S^1$ is also a covering space)

In this next section, the problem will transform to the S^1-equivariant $S(k)$-map, $g : M \longrightarrow S(k)$, and the associated maps.

4.4 Proof of Theorem (4.2)

Let F denote the fixed point set of the S^1-action on $S(k)$. F has 2^k element, $V = q^{-1}(U_0)$ has 2^k components V_y, for each $y \in F$. Each V_y is diffeomorphic to U_0 by q. U denotes the inverse image $g^{-1}(V)$, N denotes $g^{-1}(F)$. Similarly, U is the disjoint union of U_y, for $y \in F$, and $N = \bigcup_{y \in F} N_y$, where $N_y = g^{-1}(y)$, $U_y = g^{-1}(V_y)$, for $y \in F$. Note: V_y is a contractible open neighborhood of y, $(y \in F)$, $H_i(V_y) = 0$, $H_i(V) = 0$, for $i > 0$.

Consider the commutative diagrams

$$
\begin{array}{ccccc}
H_{2k-3}(M/S^1) & \xrightarrow{\beta_M} & H_{2k}(SO(M)) & \xrightarrow{\widetilde{g}_*} & H_{2k}(S(k)) \\
\downarrow{\overline{g}_*} & & \downarrow{SO(g)_*} & \nearrow \alpha_* & \\
H_{2k-3}(S(k)/S^1) & \xrightarrow{\beta_S} & H_{2k}(SO(S(k))) & &
\end{array}
\tag{4.7}
$$

where $\alpha : SO(S(k)) \longrightarrow S(k)$ is given by $\alpha(\sigma, y) = \sigma \cdot y$ (thus, $\widetilde{g}_* \circ \beta_M = \alpha_* \circ \beta_S \circ \overline{g}_*$), and

$$\textbf{(4.8)} \qquad \begin{array}{ccc}
H_{2k-3}(M/S^1) & \xrightarrow{\overline{g}_*} & H_{2k-3}(S(k)/S^1) \\
\big\downarrow & & \big\downarrow {\scriptstyle (I):\text{injective}} \\
H_{2k-3}(M/S^1, M-N/S^1) & \longrightarrow & H_{2k-3}(S(k)/S^1, S(k)-F/S^1) \\
\big\uparrow {\scriptstyle \wr (II')} & & \big\uparrow {\scriptstyle \wr (II)} \\
H_{2k-3}(U/S^1, U-N/S^1) & \longrightarrow & H_{2k-3}(V/S^1, V-F/S^1) \\
\big\downarrow & & \big\downarrow {\scriptstyle \wr (III)} \\
H_{2k-4}(U-N/S^1) & \xrightarrow{\overline{\overline{g}}_*} & H_{2k-4}(V-F/S^1) \\
\Big\| & & \Big\| \\
\bigoplus_{y \in F} H_{2k-4}(U_y - N_y/S^1) & & \bigoplus_{y \in F} H_{2k-4}(V_y - y/S^1)
\end{array}$$

Lemma (4.9):

(i) The map (I) is an injective homomorphism.

(ii) The map (II) and (II') are excision isomorphism.

(iii) The map (III) is an isomorphism.

We shall prove Lemma (4.9) later and use the results in (4.9) to finish the proof of Theorem (4.2).

By the commutativity of (4.7), if $\overline{g}_* = 0$ in $\dim(2k-3)$, then $\widetilde{g}_* \circ \beta_M = 0$.

By the commutativity of (4.8), if $\overline{\overline{g}}_* = 0$ in $\dim(2k-4)$, then $\overline{g}_* = 0$ in $\dim(2k-3)$. But, $\overline{\overline{g}}_*$ is the direct sum of $(\overline{\overline{g}}_y)_* : H_{2k-4}(U_y - N_y/S^1) \longrightarrow$

$H_{2k-4}(V_y - y/S^1)$, for $y \in F$. And, for each $y \in F$, $\overline{\overline{g}}_y$ is just a copy of $\overline{\overline{f}}$: $U_1 - Y/S^1 \longrightarrow U_0 - \xi_0/S^1$. Thus, if $\overline{\overline{f}}_*= 0$ in $\dim(2k-4)$ then $\overline{\overline{g}}_*= 0$ in $\dim(2k-4)$, and hence, $\widetilde{g}_* \circ \beta_M = 0$ which is equivalent to $\widetilde{f}_* \circ \beta_X = 0$.

Now return to Lemma (4.9), (ii) and (iii) are obvious. To prove (i), it is enough to show that $H_{2k-3}(S(k)-F/S^1)$ is trivial. In next section, we study $S(k) - F$ and prove the result (4.13) we need.

4.5 The space $S(k) - F$

Let $F_0 = \{(0,0,1),(0,0,-1)\}$, the fixed point set of S^2. Let $T = S^2 - F_0$. Then

$$S(k) - F = (T \times \prod_{k-1} S^2) \cup (S^2 \times T \times \prod_{k-2} S^2) \cup \cdots \cup (T \times \prod_{k-1} S^2).$$

Lemma (4.10): If Y is a S^1-space, then $Y \times S^1/S^1$ is homeomorphic to Y.

Proof: $f : Y \longrightarrow Y \times S^1/S^1$, $f(y) = (y,1)$, and $g : Y \times S^1/S^1 \longrightarrow Y$ by $g(y,\lambda) = \lambda^{-1} \cdot y$. Then both $f \circ g$ and $g \circ f$ are identity map.

Because $T = (0,1) \times S^1$, $T/S^1 \approx (0,1)$ and $T \times S(k-1)$ is homeomorphic to $(0,1) \times S(k-1)$.

Thus $H_i(T \times S(k-1)) = 0$, $i > 2k-2$, $H_{2k-2}(T \times S(k-1)) \approx \mathbf{Z}$ and $H_{2k-3}(T \times S(k-1)) = 0$.

Let $X_1 = T \times S(k-1)$, $X_2 = S^2 \times T \times S(k-2)$, $\cdots$, $X_i = S(i-1) \times T \times S(k-i)$, $\cdots$, $X_k = S(k-1) \times T$, $T = S^2 - \{(0,0,1),(0,0,-1)\}$. $X_i = S(i-1) \times T \times S(k-i)$, it is an open subset of $S(k)$. $X_{i_1,i_2,\cdots,i_r}$, $i_1 < i_2 < \cdots < i_r$, is the intersection of $X_{i_j, j=1,2,\cdots,r}$, that is, $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_r}$. $\{X_1, X_2, \cdots, X_k\}$ is an open covering of $S(k) - F$. Let $Y_i = X_i/S^1$, $Y = S(k)-F/S^1$. Then $\{Y_1, Y_2, \cdots, Y_k\}$ is an open covering of Y. Similarly, let $Y_{i_1,i_2,\cdots,i_r} = Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_r}$, it is equal to $X_{i_1,i_2,\cdots,i_r}/S^1$.

Homology of $Y_{i_1,i_2,\cdots,i_r}$ at dimensions $2k-r-2$ and $2k-r-3$

Without loss of generality, it is enough to understand $H_{2k-r-3}(Y_{1,2,\cdots,r})$ and $H_{2k-r-2}(Y_{1,2,\cdots,r})$.

1. $r = 1$. $H_{2k-3}(Y_1)$:

   $X_1 = T \times S(k-1)$ which has the equivariant homotopy type as $S^1 \times S(k-1)$, the map

   $$\begin{array}{rcl} S^1 \times S(k-1) & \hookrightarrow & X_1 \\ (\lambda, y) & \longrightarrow & (\lambda, 0, y), \end{array}$$

   sending S^1 into the equator of $T \subset S^2$, is an explicit equivariant homotopy equivalence. Thus $Y_1 = X_1/S^1 \simeq S^1 \times S(k-1)/S^1 \approx S(k-1)$. Thus, $H_{2k-3}(Y_i) = 0$, $\forall\ i = 1, 2, \cdots, k$.

   $H_{2k-4}(Y_1) =?$:

   Consider the embedding

   $$\psi^2 : S^1 \times S(k-2) \longrightarrow X_1,$$
   $$\psi^2 : (\lambda, y) = (\lambda a, \lambda b, y),$$

   where $a \in T$, $b \in S^2$, $(\lambda a, \lambda b, y) \in T \times S^2 \times S(k-2)$. $Im\psi^2$ is a $(2k-3)$-dim. S^1-invariant set in X_1. Similarly, we may define

   $$\psi^i : S^1 \times S(k-2) \longrightarrow X_1, 2 \le i \le k,$$
   $$\psi^i(\lambda, (y_1, y_2)) = (\lambda a, y_1, \lambda b, y_2),$$

   where $y_1 \in S(i-2)$, $y_2 \in S(k-i)$, $a \in T$, $b \in S^2$.

   $\psi^2, \psi^3, \cdots, \psi^k$ represent $k-1$ independent homology classes in $H_{2k-4}(Y_1)$. To specify the classes in $H_{2k-4}(Y_1)$, we denote by ψ_1^i, $i = 2, 3, \cdots, k$ the embeddings in Y_1.

(Note: $a \in T$ and $b \in S^2$ are two chosen points and y, y_1, y_2 denote the variable in $S(k-1)$, $S(i-2)$, $S(k-i)$, respectively)

Similarly, we have $\psi_j^i : S^1 \times S(k-2) \longrightarrow X_j$, $i = 1, 2, \cdots, j-1, j+1, j+2, \cdots, k$, embeddings in X_j, which represents $(2k-4)$-dimensional homology classes in Y_j.

2. $H_{2k-4}(Y_{1,2})$:

   Consider the embedding

$$\varphi_{12} : S^1 \times S(k-2) \longrightarrow X_{1,2} = T \times T \times S(k-2),$$
$$\varphi_{12}(\lambda, y) = (\lambda a, \lambda b, y),$$

   where a, b are constant points in T and y is a variable in $S(k-2)$. φ_{12} represents the only independent class in $H_{2k-4}(Y_{1,2})$. Similarly, embedding

$$\varphi_{i_1,i_2}(i_1 < i_2) : S^1 \times S(k-2) \longrightarrow X_{i_1,i_2}$$

   represents the only one independent homology class in Y_{i_1,i_2}.

   $H_{2k-5}(Y_{1,2})$:

   Consider the embedding

$$\psi_{1,2}^j : S^1 \times S^1 \times S(k-3) \longrightarrow X_{1,2}, j \geq 3$$
$$\psi_{12}^j(\lambda_1, \lambda_2, y_1, y_2) = (\lambda_1 a_1, \lambda_2 a_2, y_1, \lambda_1 b, y_2),$$

   where λ_1, λ_2 are variables in S^1, a_1, a_2 are constant points in T, b is a constant point in S^2, y_1 is a variable in $S(j-3)$ and y_2 is a variable in $S(k-j)$.

   Then the image of $\psi_{1,2}^j$ is an S^1-invariant $(2k-4)$-dimensional subset of $X_{1,2}$, (j could be an integer from 3 to k). These embeddings represent $(k-2)$ independent $(2k-5)$-dimension homology classes in $Y_{1,2}$.

3. $H_{2k-5}(Y_{1,2,3})$:

   Consider the embedding

$$\varphi^1_{1,2,3} : S^1 \times S^1 \times S(k-3) \longrightarrow X_{1,2,3},$$
$$\varphi^2_{1,2,3}(\lambda_1, \lambda_2, y) = (\lambda_1 a_1, \lambda_1 a_2, \lambda_2 a_3, y).$$

   Similarly, $\varphi^3_{1,2,3}(\lambda_1, \lambda_2, y) = (\lambda_1 a_1, \lambda_2 a_2, \lambda_1 a_3, y)$.

   Then, $\varphi^2_{1,2,3}$ and $\varphi^3_{1,2,3}$ represent 2 independent homology classes in $Y_{1,2,3}$.

   $H_{2k-6}(Y_{1,2,3})$:

   Consider the embedding

$$\psi^i_{1,2,3} : S^1 \times S^1 \times S^1 \times S(k-4) \longrightarrow X_{1,2,3}, i \geq 4,$$
$$\psi^i_{1,2,3}(\lambda_1, \lambda_2, \lambda_3, y_1, y_2) = (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3, y_1, \lambda_1 b, y_2),$$

   where y_1 is a variable in $S(i-4)$, y_2 is a variable in $S(k-i)$, a_1, a_2, a_3 are constant points in T, b is a constant point in S^2, $\lambda_j \in S^1$, $j = 1, 2, 3$.

   Then $\psi^i_{1,2,3}$, $i = 4, 5, \cdots, k$. represent $(k-3)$ independent homology classes in $Y_{1,2,3}$.

4. $(H_{2k-r-2}(Y_{i_1,i_2,\cdots,i_r}))$

   Simply: $H_{2k-r-2}(Y_{1,2,\cdots,r})$,

$$\varphi^2_{1,2,\cdots,r} : (\prod_{r=1} S^1) \times S(k-r) \longrightarrow X_{1,2,\cdots,r},$$
$$\varphi^2_{1,2,\cdots,r}(\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, y) = (\lambda_1 a_1, \lambda_1 a_2, \lambda_2 a_3, \lambda_3 a_4, \cdots, \lambda_{r-1} a_r, y),$$

   λ_i, $i = 1, 2, \cdots, r-1$ are variables in S^1, $a_1, a_2, \cdots, a_r$ are constant points in T, y is a variable in $S(k-r)$.

$$\varphi^3_{1,2,\cdots,r}(\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, y)$$

$$
\begin{aligned}
&= (\lambda_1 a_1, \lambda_2 a_2, \lambda_1 a_3, \lambda_3 a_4, \lambda_4 a_5, \cdots, \lambda_{r-1} a_r, y) \\
&\vdots \\
&\varphi^r_{1,2,\cdots,r}(\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, y) \\
&= (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3, \cdots \lambda_{r-1} a_{r-1}, \lambda_1 a_r, y)
\end{aligned}
$$

(Remark: $\varphi^i_{1,2,\cdots,r}(\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, y) = (\lambda_1 a_1,\ \cdots,\ \lambda_1 a_i,\ \cdots\ y))$

$\varphi^2_{1,2,\cdots,r},\ \varphi^3_{1,2,\cdots,r},\ \cdots,\ \varphi^r_{1,2,\cdots,r}$ represent $(r-1)$ independent homology classes in $Y_{1,2,\cdots,r}$.

(Note: $Y_{1,2,\cdots,r}$ is homotopy equivalence to $(\prod\limits_{r-1} S^1) \times (\prod\limits_{k-r} S^2)$, rank $H_{2k-r-2}(Y_{1,2,\cdot,r}) = r-1$.)

$H_{2k-r-3}(Y_{i_1,i_2,\cdots,i_r})$, $1 \le i_1 < i_2 < \cdots < i_r \le k$:

For convenience, consider $i_1 = 1, i_2 = 2, \cdots, i_r = r$.

Let

$$
\begin{aligned}
&\psi^i_{1,2,\cdots,r} : \prod_r S^1 \times S(k-r-1) \longrightarrow X_{1,2,\cdots,r} \text{ be defined by :} \\
&\psi^i_{1,2,\cdots,r}(\lambda_1, \lambda_2, \cdots, \lambda_r, y_1, y_2) = (\lambda_1 a_1, \lambda_2, a_2, \cdots, \lambda_r a_r, y_1, \lambda_1 b, y_2)
\end{aligned}
$$

where $k \ge i \ge r+1$, $y_1 \in S(i-r-1)$, $y_2 \in S(k-i)$, $a_1, a_2, \cdots, a_r$ are constant points in T, and b is a constant point in S^2.

$\psi^{r+1}_{1,2,\cdots,r},\ \psi^{r+2}_{1,2,\cdots,r},\ \cdots,\ \psi^k_{1,2,\cdots,r}$ represent $(k-r)$ independent homology classes in $Y_{1,2,\cdots,r}$.

Note: $Y_{i_1,i_2,\cdots,i_r}$ is homology equivalence to $\prod\limits_{r-1} S^1 \times \prod\limits_{k-r} S^2$, thus, $H_{2k-r-3}(Y_{i_1,i_2,\cdots,i_r})$ has rank $(k-r)$.

Let $\overline{G}_r$ be the direct sum of $H_{2k-r-3}(Y_{i_1,i_2,\cdots,i_r})$, for all $(i_1, i_2, \cdots, i_r)$ satisfying $1 \le i_1 < i_2 < \cdots < i_r \le k$. And $G_{i_1,i_2,\cdots,i_{r+1}}$ be the subgroup of $\overline{G}_r$, generated by the $(r+1)$ elements $\psi^{i_1}_{i_2,i_3,\cdots,i_{r+1}},\ \psi^{i_2}_{i_1,i_3,\cdots,i_{r+1}},\ \cdots,\ \psi^{i_{r+1}}_{i_1,i_2,\cdots,i_r}$, for any $(i_1, i_2, \cdots, i_{r+1})$ satisfying $1 \le i_1 < i_2 < \cdots < i_{r+1} \le k$.

Thus,

$$\overline{G}_r = \bigoplus_{1\le i_1<i_2<\cdots<i_{r+1}\le k} G_{i_1,i_2,\cdots,i_{r+1}}.$$

Let

$$\rho^j_{i_1,i_2,\cdots,i_r} : Y_{i_1,i_2,\cdots,i_r} \longrightarrow Y_{i_1,\cdots,\widehat{i_j},\cdots,i_r}$$

be the inclusion, $1 \le j \le r$, and

$$\eta_{i_1,i_2,\cdots,i_r} : H_{2k-r-2}(Y_{i_1,i_2,\cdots,i_r}) \longrightarrow \bigoplus_{j=1}^{r} H_{2k-r-2}(Y_{i_1,i_2,\cdots,\widehat{i_j},\cdots,i_r}),$$

be the homomorphism

$$\eta_{i_1,i_2,\cdots,i_r}(\alpha) = ((\rho^1_{i_1,i_2,\cdots,i_r})_*(\alpha), \cdots, (\rho^r_{i_1,i_2,\cdots,i_r})_*(\alpha)).$$

Proposition (4.11): $\eta_{i_1,i_2,\cdots,i_r}$ sends $H_{2k-r-2}(Y_{i_1,i_2,\cdots,i_r})$ injectively into $G_{i_1,i_2,\cdots,i_r}$, which is a subgroup of $\overline{G}_{r-1}$.

Let H_r be the direct sum of homology groups $H_{2k-r-2}(Y_{i_1,i_2,\cdots,i_r})$, for all $(i_1, i_2, \cdots, i_r)$ satisfying $1 \le i_1 < i_2 < \cdots < i_r \le k$. And $\zeta_r : H_r \longrightarrow \overline{G}_{r-1}$ be the direct sum of $\eta_{i_1,i_2,\cdots,i_r}$, that is, $\zeta_r((\alpha_{I_1}, \alpha_{I_2}, \cdots, \alpha_{I_l})) = \sum_{i=1}^{l} \eta_{I_i}(\alpha_{I_i})$, where I_i is some $(i_1, i_2, \cdots, i_r)$. Thus, we conclude

Corollary (4.12): $\zeta_r : H_r \longrightarrow \overline{G}_{r-1}$ is injective.

Proof of Proposition (4.11): It is enough to check the following facts:

(i) $\rho^i_{1,2,\cdots,r} \circ \varphi^i_{1,2,\cdot,r} = \psi^i_{1,2,\cdots,\widehat{i},\cdots,r}$, for $i \ge 2$.

(ii) $\rho^1_{1,2,\cdots,r} \circ \varphi^i_{1,2,\cdots,r} = \psi^1_{2,3,\cdots,r}$, for $i \ge 2$. (at least, represent the same homology class)

(iii) $\rho^j_{1,2,\cdots,r} \circ \varphi^i_{1,2,\cdots,r}$ is homotopic to a map which has image with less 1 dimension in $Y_{1,2,\cdots,\widehat{j},\cdots,r}$, for $i \ge 2$ and $j \ne 1, i$.

Thus, $\eta_{1,2,\cdots,r}(\varphi^i_{1,2,\cdots,r}) = (1,0,\cdots,1,0,\cdots)$ in $G_{1,2,\cdots,r}$, for $i \geq 2$.

$Im(\eta_{1,2,\cdots,r})$ has rank at least $r-1$. But, rank $(H_{2k-r-2}(Y_{1,2,\cdots,r})) = r-1$.

Theorem (4.13): $H_{2k-3}(S(k) - F/S^1) = 0$.

Proof: $S(k)-F/S^1 = Y_1 \cup Y_2 \cup \cdots \cup Y_k$ in $S(k)/S^1$. By standard spectral sequence argument, the following facts will implies $H_{2k-3}(Y_1 \cup Y_2 \cup \cdots \cup Y_k) = 0$.

(i) $H_{2k-3}(Y_i) = 0$, for $i = 1, 2, \cdots, k$.

(ii) $\bigoplus_{1\le i<j\le k} H_{2k-4}(Y_i \cap Y_j) \xrightarrow{\zeta_2} \bigoplus_{1\le i\le k} H_{2k-4}(Y_i)$ is injective.

(iii) $\bigoplus_{1\le i\le j<l\le k} H_{2k-5}(Y_i \cap Y_j \cap Y_l) \xrightarrow{\zeta_3} \bigoplus_{1\le i<j\le k} H_{2k-5}(Y_i \cap Y_j)$ is injective.

(iv) , $\cdots$, etc.

where ζ_r is the direct sum of the natural maps, for example $r=3$, $\eta_{i,j,l}$ is the composite map

$$\begin{array}{ccc} H_*(Y_i \cap Y_j \cap Y_l) & \longrightarrow & H_*(Y_i \cap Y_j) \bigoplus H_*(Y_i \cap Y_l) \bigoplus H_*(Y_j \cap Y_l) \\ & & \downarrow \\ & & \bigoplus_{1\le a<b\le k} H_*(Y_a \cap Y_b) \end{array}$$

and $\zeta_3 = \sum_{1\le i<j<l\le k} \eta_{i,j,l}$.

§5 Normal bundle of the fixed point set in $\mathcal{W}_0$

$\mathcal{W}_0(n)$ is a complicated C. W. complex, it is not clear that a subspace in $\mathcal{W}_0(n)$ has a meaningful normal vector bundle. But, the fixed point set $\mathcal{H}(n)$ does have a $2n$-dimensional complex vector bundle as its normal in $\mathcal{W}_0(n)$.

5.1 Inhomogeneous coordinate of $P(k)$

$\rho : \mathbb{C} = \mathbb{R}^2 \longrightarrow \mathbb{RP}^2$, $\rho(x, y) = [(x, y, 1)]$, maps $\mathbb{C}$ homeomorphically onto an open set of $\mathbb{RP}^2$.

$\rho_k : \mathbb{C}^k \longrightarrow P(k)$, $P_k(\lambda_1, \lambda_2, \cdots, \lambda_k) = (\rho(\lambda_1), \rho(\lambda_2), \cdots, \rho(\lambda_k))$, maps $\mathbb{C}^k$ homeomorphically onto an open dense set of $P(k) = \prod_k \mathbb{RP}^2$. ξ_0 denotes the fixed point of the S^1-action on $P(k)$ which is defined in section 4.1. $\rho_k(\mathbb{C}^k)$ is a contractible open neighborhood of ξ_0.

Let U_k denote the open neighborhood $\rho_k(\mathbb{C}^k)$ of ξ_0.

Suppose Γ is a knot graph. As in section 3.1, $W_0(\Gamma)$ is the configuration space of infinitesimal knot graph on the line l_0 of z-axis, $\Psi : W_0(\Gamma) \longrightarrow P(k)$ is a canonical $P(k)$-structure.

Suppose $f : \Gamma \longrightarrow \Gamma'$ is an equivalence of knot graphs and Γ' is a knot graph on l_0. For any $v \in V(\Gamma)$, $f(v)$ is a point of $\mathbb{R}^3$. Identify $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$, and write the function $f : V(\Gamma) \longrightarrow \mathbb{R}^3$ as (g, h), $g : V(\Gamma) \longrightarrow \mathbb{C}$ and $h : V(\Gamma) \longrightarrow \mathbb{R}$. Thus, the function g, satisfying $g|_{V_0(\Gamma)} = 0$, is called a ground function of Γ and the function h, preserving the linear order of base points in $V_0(\Gamma)$, is called a height function of Γ. A ground function g of Γ together with a height function h of Γ forms a knot graph on l_0. But a height function could be thought as a knot graph whose vertices are all on the line l_0.

Suppose Γ has s inner vertices. $C(\Gamma, l_0)$ is "almost" an s-dimensional

complex vector bundle over the space of all height function, and $W_0(\Gamma)$ is also "almost" an s-dimensional complex vector bundle over the space { all height function of Γ} / translation and dilation relation. The relation is the following:

h_1, h_2 are two height function of Γ,

$h_1 \overset{\text{t.d.}}{\sim} h_2$, if there exist $\lambda > 0$ and real number t such that $h_1(v) = \lambda h_2(v) + t$.

But, we are more interested in the knot graph $f = (g, h)$ on l_0 which is in $\Psi^{-1}(U_k)$. Why?

Suppose Γ has edges $e_1 = \{v_1, w_1\}$, $e_2 = \{v_2, w_2\}$, $\cdots$, $e_k = \{v_k, w_k\}$. Then

$$\begin{aligned}&\Psi(f)(\text{ more formally } \Psi(f, f(\Gamma)))\\&= ([f(v_1) - f(w_1)], [f(v_2) - f(w_2)], \cdots, [f(v_k) - f(w_k)]).\end{aligned}$$

When $\Psi(f)$ is in $U_k = \rho_k(\mathbb{C}^k)$,

$$\rho_k^{-1}(\Psi(f)) = \left(\frac{g(v_1) - g(w_1)}{h(v_1) - h(w_1)}, \frac{g(v_2) - g(w_2)}{h(v_2) - h(w_2)}, \cdots, \frac{g(v_k) - g(w_k)}{h(v_k) - h(w_k)}\right).$$

Thus, in the coordinate system $\rho_k : \mathbb{C}^k \longrightarrow P(k)$, the $P(k)$-structure $\Psi : W_0(\Gamma) \longrightarrow P(k)$ is an injective fibre-wise linear map.

To formulate the statement above precisely, let us consider the following definitions.

Definition (5.1): Suppose $g_i : V(\Gamma) \longrightarrow \mathbb{C}$ are ground functions and $h_i : V(\Gamma) \longrightarrow \mathbb{R}$ are height functions, $i = 1, 2$. (g_1, h_1) is equivalent to (g_2, h_2) under the translation and dilation relation, if there exist real number λ and t, $\lambda > 0$, such that $g_1(v) = \lambda g_2(v)$ and $h_1(v) = \lambda h_2(v) + t$ for all $v \in V(\Gamma)$.

Thus, $W_0(\Gamma)$ is contained in and "almost" equal to the space $\{(g, h) : g$ is a ground function of Γ and h is a height function of $\Gamma\}$ / T.D. relation.

(Recall: $g|_{V_0(\Gamma)} = 0$ and h preserves the linear order of base points, that is, if $x_1 < x_2$ in $V_0(\Gamma)$, then $h(x_1) < h(x_2)$.)

Definition (5.2): Suppose Γ is a knot graph. $D(\Gamma)$ denotes the subspace of $W_0(\Gamma)$, $\{(g,h) \in W_0(\Gamma) : h(v_i) \neq h(w_i)$, for all $e_i = \{v_i, w_i\}$ in $E(\Gamma)\}$ / T.D. relation, and $H(\Gamma) = \{h : V(\Gamma) \longrightarrow \mathbb{R} : h(x) < h(x')$, for any two base points $x < x'$ in $V_0(\Gamma)$, and $h(v_i) \neq h(w_i)$, for all edge $e_i = \{v_i, w_i\}$ in $E(\Gamma)\}$ / T.D. relation. Then $0 \times H(\Gamma) = \{(0,h) : h \in H(\Gamma)\}$ is a subspace of $D(\Gamma)$, also, a subspace of $W_0(\Gamma)$.

Proposition (5.3):

(i) $D(\Gamma) = \Psi^{-1}(U_k)$

(ii) $0 \times H(\Gamma) = \Psi^{-1}(\xi_0)$, it is the set of all fixed points of S^1-action on $W_0(\Gamma)$ given in section 3.3.

(iii) Assume Γ has s inner vertices. $D(\Gamma)$ is an s-dimensional complex vector bundle over $H(\Gamma)$.
(Proof is straightforward and is omitted)

For convenience, we identify $H(\Gamma)$ as $0 \times H(\Gamma)$. Thus, a height function is also a knot graph on l_0. But, there is no way to think a ground function as a knot graph on l_0.

Let $D(\Gamma)_h = \{(g,h) : (g,h) \in D(\Gamma)\}$, it is the s-dimensional complex vector space over h. The vector bundle $D(\Gamma)$ over $H(\Gamma)$ is trivial, but there is no canonical basis.

Proposition (5.4): Suppose Γ is a knot graph and h is a height function in $H(\Gamma)$. Then, the restriction Ψ_h of $\rho_k^{-1} \circ \Psi$ to $D(\Gamma)_h$ is an injective linear map from $D(\Gamma)_h$ to $\mathbb{C}^k$. (Note: order $(\Gamma) = k - s$ is always positive.)

(Rigorous consideration of injectivity is in section 6.2 and 6.3 .)

Question: Could $\Psi_h : D(\Gamma)_h \longrightarrow \mathbb{C}^k$ continuously extend to the boundary of $H(\Gamma)$? (What is the boundary of $H(\Gamma)$?)

To answer the question, there are still many works to do. For convenience, we abuse the notations: to denote $\rho_k^{-1} \circ \Psi$ also by Ψ. Then

$$\Psi(g, h) = \left(\frac{g}{h}(e_1), \frac{g}{h}(e_2), \cdots, \frac{g}{h}(e_k) \right),$$

where

$$\frac{g}{h}(e_i) = \frac{g(v_i) - g(w_i)}{h(v_i) - h(w_i)}, \text{ for } e_i = \{v_i, w_i\}.$$

And the fibrewise linear map associated with $W_0(\Gamma) \times P(r)$ is $\Psi \times id_{\mathbb{C}^r} : D(\Gamma) \times \mathbb{C}^r \longrightarrow \mathbb{C}^{k+r}$. And we also denote the stable fibrewise linear map by Ψ. For any σ in Σ_{k+r}, $\sigma : \mathbb{C}^{k+r} \longrightarrow \mathbb{C}^{k+r}$, $\sigma(\lambda_1, \lambda_2, \cdots, \lambda_{k+r}) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \cdots, \lambda_{\sigma(k+r)})$, is a linear isomorphism. Thus, $\Sigma_{k+r} \cdot \Psi$ will consists of all the fibrewise linear maps associated with all the canonical $P(k+r)$-structure of $W_0(\Gamma) \times P(r)$. Similar to the construction of $\mathcal{W}_0(n)$, let $\mathcal{D}(n)$ be the disjoint union of $D(\Gamma) \times \mathbb{C}^{3n-k} \times \Sigma_{3n} \cdot \Psi$, for all normal knot graphs Γ with order n, and let $\overline{\mathcal{H}}(n)$ be the disjoint union of $H(\Gamma) \times \Sigma_{3n} \cdot \Psi$, for all normal knot graph Γ with order n (note: the function Ψ, multiplied with $H(\Gamma)$, is the same as the function Ψ, multiplied with $D(\Gamma) \times \mathbb{C}^{3n-k}$). Then $\overline{\mathcal{D}}(n)$ is a $2n$-dimensional complex vector bundle over $\overline{\mathcal{H}}(n)$. Now, we need identify the components in $\overline{\mathcal{D}}(n)$ and $\overline{\mathcal{H}}(n)$ along their boundaries.

Their boundaries are similar to the boundaries of configuration spaces $C(\Gamma)$. With the boundaries, $D(\Gamma)$ will be substituted by its base-compactification and so are $H(\Gamma)$.

5.2 Boundary of $W_0(\Gamma)$

There is a special identification in the boundary of $W_0(\Gamma)$ we should do prior to the other identifications.

Suppose $A(\Gamma)$ is the disjoint union of connected components $A_1(\Gamma), A_2(\Gamma), \cdots$, and $A_m(\Gamma)$, where A is the disjoint union of $A_1, A_2, \cdots$, and A_m. The associated extended translation and dilation relation is called a special extended translation and dilation relation, or simply, a S.E.T.D. relation associated to (A, Γ). (see (3.5) in section 3.1)

Notation (5.5):

(i) $Q(A(\Gamma))$ is the space $W_0(A(\Gamma))$ quotiented by the S.E.T.D. relation associated with (A, Γ).

(ii) $Q(\Gamma; A) = W_0(\Gamma/A) \times Q(A(\Gamma))$.

Remark: $Q(A(\Gamma))$ is exactly a subspace of $W_0(A_1(\Gamma)) \times W_0(A_2(\Gamma)) \times \cdots \times W_0(A_m(\Gamma))$, and $Q(\Gamma; A)$ is the more qualified boundary than $W_0(\Gamma; A)$, because of the following proposition.

Proposition (5.6): Outside some odd part, $W_0(\Gamma) \bigcup_A Q(\Gamma; A)$ with the natural topology is an s-dimensional complex vector bundle over its fixed point set $\mathrm{Fix}\,(W_0(\Gamma) \bigcup_A Q(\Gamma; A))$. (In this proposition, $W_0(\Gamma)$ return to its original space without compactification and $W_0(\Gamma) \bigcup_A Q(\Gamma; A)$ is a locally compact space containing $W_0(\Gamma)$.)

Proof: Let $\mathrm{Fix}\,(\Gamma)$ denote the space $\mathrm{Fix}\,(W_0(\Gamma) \bigcup_A Q(\Gamma; A))$.

We first define the bundle projection

$$\pi : (W_0(\Gamma) \bigcup_A Q(\Gamma; A) \longrightarrow \mathrm{Fix}\,(\Gamma),$$
$$(g, h) \in W_0(\Gamma), \pi(g, h) = (0, h),$$

similarly, for

$$f_0 = (g_0, h_0) \in W_0(\Gamma/A)$$

and

$$f_i = (g_i, h_i) \in W_0(A_i(\Gamma)),\ i = 1, 2, \cdots, m,$$
$$\alpha = (f_0, f_1, f_2, \cdots, f_m) \in Q(\Gamma; A),$$
$$\pi(\alpha) = ((0, h_0), (0, h_1), \cdots, (0, h_m)).$$

Then $\beta = ((0, h_0), (0, h_1), \cdots, (0, h_m))$ is in $\mathrm{Fix}\,(\Gamma)$.

But $\pi^{-1}(\beta)$ may contains more elements other than that in $W_0(\Gamma; A)$. If $f = (g, h) \in W_0(\Gamma)$ such that $g|_{A_i}$ is constant, for each $i = 1, 2, \cdots, m$ and $h = h_0$, then $\pi(f, (g_1, h_1), \cdots, (g_m, h_m))$ is also equal to β. Although $\gamma = (f, (g_1, h_1), \cdots, (g_m, h_m))$ is not in the interior of a codimension 1 boundary $W_0(\Gamma; A')$, for some A', γ is in the boundary associated to the sets $(A_1, A_2, \cdots, A_m)$. Thus, $\pi^{-1}(\beta)$ is the vector space $G = \{((g, h_0), (g_1, h_1), \cdots, (g_m, h_m)) : g : V(\Gamma) \longrightarrow \mathbb{C}, g|_{A_i}$ is constant, for each $i = 1, 2, \cdots, m$, and $g_i : A_i \longrightarrow \mathbb{C}, i = 1, 2, \cdots, m\}$ quotiented by the following relations:

$\gamma = ((g, h_0), (g_1, h_1), \cdots, (g_m, h_m))$ and $\gamma' = ((g', h_0), (g'_1, h_1), \cdots, (g'_m, h_m))$ are two elements in G,

$\gamma \sim \gamma'$, if there exist complex numbers $c_1, c_2, \cdots, c_m$ such that $g_i = g'_i + c_i$, for each $i = 1, 2, \cdot, m$.

Thus, $\pi^{-1}(\beta)$ has a natural vector space structure and has s dimension.

And the set $G'(h_0) = \{(g, h_0) \in W_0(\Gamma)$, for some i, $g|_{A_i}$ is not constant $\}$ which contains elements in $\pi^{-1}\ ((0, h_0))$ but not $\pi^{-1}\ ((0, h_0), (0, h_1), \cdots, (0, h_m))$, for any $h_1, h_2, \cdots, h_m$, $h_i : A_i \longrightarrow \mathbb{R}$, it is the odd part of $W_0(\Gamma) \bigcup\limits_A Q(\Gamma; A)$ in the fibre over β. This proves (5.6).

Remarks on the proof of (5.6):

(i) In $W_0(\Gamma) \bigcup\limits_A W_0(\Gamma; A)$, if $g(A_i) = g(A_j)$, for some $i \neq j$, there is an unnecessary compactification in $W_0(\Gamma; A)$, which destroies the natural vector space structure.

(ii) Consider the following commutative diagram

$$\begin{array}{ccc} Q(\Gamma; A) & \xrightarrow{q'} & W_0(\Gamma/A) \\ \big\downarrow \pi & & \big\downarrow \pi' \\ \mathrm{Fix}\,(Q(\Gamma; A)) & \xrightarrow{q} & \mathrm{Fix}\,(W_0(\Gamma/A)) \\ \beta & \longrightarrow & h_0 \end{array}$$

where $q(h_0, h_1, \cdots, h_m) = h_0$ and $q'(f_0, f_1, f_2, \cdots, f_m) = f_0$. Although q is onto obviously, $\pi^{-1}(h_0)$ is larger than $\pi^{-1}(q^{-1}(h_0)) = q'^{-1}(\pi'^{-1}(h_0))$. And the odd part $G'(h_0)$ is equal to $\pi^{-1}(h_0) - \pi^{-1}(q^{-1}(h_0))$, which is contained in $W_0(\Gamma)$ and not in $Q(\Gamma; A)$.

(iii) Suppose $\Psi : W_0(\Gamma) \bigcup_A Q(\Gamma; A) \longrightarrow P(k)$ is a canonical $P(k)$-structure. The complement of $\Psi^{-1}(U_k)$ is exactly the odd part.

(iv) With some abuse of notation, we may denote $W_0(\Gamma) \bigcup_A Q(\Gamma; A)$ by $\overline{W}_0(\Gamma)$, the odd part of $\overline{W}_0(\Gamma)$ by $Odd(W_0(\Gamma))$, and $\overline{W}_0(\Gamma) - Odd(W_0(\Gamma))$ by $D(\Gamma)$. Then $\Psi : D(\Gamma) \longrightarrow \mathbb{C}^k$ is the fibre-wise injective linear map, which has a simple form as follows:

(Case 1) Interior of $W_0(\Gamma)$.

$$f = (g, h),$$
$$\Psi(f) = \left(\frac{g}{h}(e_1), \frac{g}{h}(e_2), \cdots, \frac{g}{h}(e_k)\right)$$

(Case 2) Codimension 1 boundary ($A(\Gamma)$ has only one connected component).

$\alpha = (f_0, f_1)$, $f_0 = (g_0, h_0)$, $g_0|_A$ and $h_0|_A$ are constant, $f_1 = (g_1, h_1)$, $g_1 : A \longrightarrow \mathbb{C}$, $h_1 : A \longrightarrow \mathbb{R}$. Assume $e_1, e_2, \cdots, e_r$ are in $A(\Gamma)$.

$$\Psi(\alpha) = \left(\frac{g_1}{h_1}(e_1), \frac{g_1}{h_1}(e_2), \cdots, \frac{g_1}{h_1}(e_r), \frac{g_0}{h_0}(e_{r+1}), \cdots, \frac{g_0}{h_0}(e_k)\right)$$

(Case 3) $A(\Gamma)$ has more than 1 connected components.

Ψ has similar form as in (case 2).

(Note: (i) if $e = \{v, w\}$, $\frac{g}{h} = \frac{g(v)-g(w)}{h(v)-h(w)}$, (ii) Ψ is, in fact, $\rho_k^{-1} \circ \Psi$.)

5.3 Identification maps of vector bundles

A vector bundle map is a (bundle) map between two vector bundles, which is fibrewise linear isomorphism. Now, we consider all the "identification" vector bundle maps come from the identification map in section 1.3 except the identification maps of type 0, which has been done in the last section (5.2).

5.3.1 Identification map of type II

Suppose $A(\Gamma)$ has a bivalent inner vertice v, $e_1 = \{v, w, \}$ and $e_2 = \{v, w_2\}$ are the two edge connecting to v. Without loss of generality, we may assume $A(\Gamma)$ is connected. Thus $Q(\Gamma; A) = W_0(\Gamma; A)$.

Assume $f = (g, h) \in Q(\Gamma; A)$. Then, the identification map τ_2 sending f to $\tau_2(f) = (\tau_2'(g), \tau''{}_2(h))$

$$\tau_2(f)(v) = f(w_1) + f(w_2) - f(v)$$
$$\tau(f)(v') = f(v'), \quad \text{if} \quad v' \neq v.$$

Thus,

$$\tau_2'(g)(v) = g(w_1) + g(w_2) - g(v),$$
$$\tau_2'(g)(v') = g(v'), \quad \text{if} \ \ v' \neq v;$$

and

$$\tau_2''(h)(v) = h(w_1) + h(w_2) - h(v),$$
$$\tau_2''(h)(v') = h(v'), \quad \text{if} \ \ v' \neq v.$$

Therefore, $\tau_2' : D(\Gamma; A)_h \longrightarrow D(\Gamma; A)_{\tau_2''(h)}$ is a linear isomorphism of fibres, where

$$\begin{aligned} D(\Gamma; A) \quad &= \Psi^{-1}(U_k) \cap Q(\Gamma; A), \\ \text{also} \quad &= D(\Gamma) \cap Q(\Gamma; A). \end{aligned}$$

5.3.2 Identification map of type III

Suppose $A = \{v, w\}$ is an edge of Γ, and v is an inner vertice. $\tau_3 : W_0(\Gamma; A) \longrightarrow W_0(\Gamma/A) \times \mathbb{RP}^2$ is the identification map defined in section 1.3.4.

Assume $\alpha = (f_0, f_1) \in W_0(\Gamma/A) \times Q(A(\Gamma))$, $f_1 = (g_1, h_1)$, $g_1 : A \longrightarrow \mathbb{C}$, $h_1 : A \longrightarrow \mathbb{R}$, $f_0 \in W_0(\Gamma/A)$. Then

$$\tau_3(\alpha) = \left(f_0, \rho \left(\frac{g_1(v) - g_1(w)}{h_1(v) - h_1(w)} \right) \right),$$

$\rho : \mathbb{C} \longrightarrow \mathbb{RP}^2$ is defined in section 5.1. If $f_0 = (g_0, h_0)$, $g_0 : V(\Gamma/A) \longrightarrow \mathbb{C}$, $h_0 : V(\Gamma/A) \longrightarrow \mathbb{R}$. Then

$$\tau_3 : D(\Gamma; A)_{(h_0, h_1)} \longrightarrow D(\Gamma/A)_{h_0} \times \mathbb{C}$$

is the following

$$\tau_3((g_0, h_0), (g_1, h_1)) = \left((g_0, h_0), \frac{g_1(v) - g_1(w)}{h_1(v) - h_1(w)} \right),$$

which is an linear isomorphism.

(Note: when $\alpha \in D(\Gamma; a) \subset \Psi^{-1}(U_k)$, $h_1(v) - h_1(w)$ cannot be zero.)

5.3.3 Identification map of type IV

In the case, τ_4 is obviously a linear isomorphism.

5.3.4 Identification of type I

(i) Alternating method:

Suppose Γ is a normal knot graph, $|A| \geq 3$, $A(\Gamma)$ has a univalent inner vertice v, $e = \{v, v_1\}$ is the unique edge connecting to v.

(Case 1) v_1 is also a univalent inner vertice in $A(\Gamma)$. Then $A(\Gamma)$ is disconnected. The special translation and dilation relation for $Q(\Gamma; A)$ has reduced $Q(\Gamma; A)$ to lower dimensional boundary.

(Case 2) v_1 is a bivalent inner vertice in $A(\Gamma)$. Then we apply the identification map of type II.

(Case 3) v_1 is a trivalent inner vertice in $A(\Gamma)$, that is, there exist edges $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_1, w_2\}$ connecting to v_1. Considering the following map, which was first given by Bott and Taubes [5]:

$$\begin{aligned}
&\tau : Q(A(\Gamma)) \longrightarrow Q(A(\Gamma)) \\
&f : A \longrightarrow \mathbb{C} \times \mathbb{R}, \ f \in Q(A(\Gamma)). \\
&\tau(f)(v) = f(w_1) + f(w_2) - f(v) \\
&\tau(f)(v) = f(w_1) + f(w_2) - f(v) \\
&\tau(f)(v_1) = f(w_1) + f(w_2) - f(v_1) \\
&\tau(f)(v') = f(v'), \quad \text{if} \ \ v' \neq v \text{ and } v' \neq v_1.
\end{aligned}$$

Thus, we have an identification map similar to the map of type II. Similarly, we have the linear isomorphism of fibres. This identification is said to be of the type Y.

(Case 4) v_1 is a base points in Γ. We may assume Γ has univalent base point only. And, it is similar to the (Case 1).

(ii) Original method:

Following the assumption in (i), v is a univalent inner vertice of $A(\Gamma)$ and $e = \{v, v_1\}$ is an edge in $A(\Gamma)$.

For convenience, assume $A(\Gamma)$ is connected, then $Q(A(\Gamma)) = W_0(A(\Gamma))$.

In section 1.3.2, we have the identification map $\tau_1 : W_0(A(\Gamma)) \longrightarrow W_0(A(\Gamma))$: for $f : A \longrightarrow \mathbb{C} \times \mathbb{R}$, let $f_1 = f|_{A-\{v\}}$ and $||f_1|| = \max\{|f_1(w) - f(w')|,\ w, w' \in A - \{v\}\}$,

$$\begin{aligned} &\tau_1(f)(v) = f(v_1) + 2||f_1||\frac{f(v) - f(v_1)}{|f(v) - f(v_1)|}, \\ &\tau_1(f)(w) = f(w), \quad \text{for } w \neq v. \end{aligned}$$

It is easy to see that τ_1 is S^1-equivariant. Thus, τ_1 sends the fixed points to fixed points. Write f as (g, h), $\pi(f) = h$. But $\pi(\tau_1(f))$ may not be $\tau_1(\pi(f))$. Let $\tau_1' : D(\Gamma; A) \longrightarrow D(\Gamma; A)$ be the map defined as follows:

$$\begin{aligned} &\alpha = ((g_0, h_0), (g, h)) \in D(\Gamma; A), \\ &g_0 : \Gamma/A \longrightarrow \mathbb{C},\ h_0 : \Gamma/A \longrightarrow \mathbb{R}, \\ &g : A \longrightarrow \mathbb{C},\ h_0 : A \longrightarrow \mathbb{R}, \end{aligned}$$

At first, let $g_1 = g|_{A-\{v\}}$, $h_1 = h|_{A-\{v\}}$.

$$\begin{aligned} &\tau_1'(g)(v) = g(v_1) + 2||h_1||\frac{g(v) - g(v_1)}{|h(v) - h(v_1)|}, \\ &\tau_1'(g)(w) = g(w), \quad \text{for } w \neq v, \\ &\tau_1'(h)(v) = h(v_1) + 2||h_1||\frac{h(v) - h(v_1)}{|h(v) - h(v_1)|}, \\ &\tau_1'(h)(w) = h(w), \quad \text{for } w \neq v, \end{aligned}$$

and $\tau_1'(\alpha) = ((g_0, h_0), (\tau_1'(g), \tau_1'(h))$. It is easy to show that

(i) when $\alpha \in \mathrm{Fix}\,(W_0(\Gamma; A))$, $\tau_1'(\alpha) = \tau_1(\alpha)$.

(ii) τ_1' is a vector bundle map.

(iii) τ_1' is the differential of τ_1 at the fixed point set.

(iv) $\Psi(\tau_1(\alpha)) = \Psi(\tau_1'(\alpha)) = \Psi(\alpha)$.

And the most important fact is that the two identification maps $\tau_1|_{D(\Gamma;A)}$ and τ_1' give the same result. Thus, we may use τ_1' instead of τ_1, when we consider only the part $D(\Gamma; A)$, that is, $\Psi^{-1}(U_k) \cap Q(\Gamma; A)$.

Remark: The alternating method for the "univalent inner vertice" identification is just reducing the type I to the other types. The merit of this method is that the identification maps for $\overline{W_0(\Gamma)}$ ($= W_0(\Gamma) \bigcup_A Q(\Gamma; A)$) are all local diffeomorphisms for both vector bundles and base spaces (the fixed point set of $Q(\Gamma; A)$)

First kind: $(|A| \geq 3)$
(Type V): $\tau_2 : Q(\Gamma; A) \longrightarrow Q(\Gamma; A)$
(Type Y): $\tau : Q(\Gamma; A) \longrightarrow Q(\Gamma; A)$

Second kind: $(|A| = 2)$
$\tau_3, \tau_4 : Q(\Gamma; A) \longrightarrow W_0(\Gamma/A) \times P(A)$.

Furthermore, the two different type identification maps of first kind are P-orientation reversing involutions. This is a crucial point in the degree theory.

5.4 Conclusion

We continue the notations $\overline{\mathcal{D}}(n)$ and $\overline{\mathcal{H}}(n)$ in section 5.1. Let $\mathcal{D}(n)$ be the quotient space of $\overline{\mathcal{D}(n)}$ by the identification maps of vector bundles given in lase section (5.3), and the fixed point set $\mathcal{H}(n)$ is also a quotient space of $\overline{\mathcal{H}}(n)$ by the related identification maps. Then $\mathcal{D}(n)$ is a $2n$-dimensional complex vector bundle over $\mathcal{H}(n)$.

Overall Conclusion

First, we state a few results:

$$\textbf{(5.7)} \qquad \begin{aligned} & H_{6n}(\mathcal{W}(n), \Psi(\mathcal{O}_{3n})) \\ & \approx H_{6n-2}(\mathcal{W}_0(n), \Psi^{\#}(\mathcal{O}_{3n})) \\ & \approx H_{6n-3}(\mathcal{W}_0(n)/S^1, \overline{\Psi}^{\#}(\overline{\mathcal{O}}_{3n})) \end{aligned}$$

(The proof is somewhat straightforward.)

Thus, the homomorphism

$\beta : H_{6n-3}(\mathcal{W}_0(n)/S^1, \overline{\Psi}^{\#}(\overline{\mathcal{O}}_{3n})) \longrightarrow H_{6n}(\mathcal{W}_0(n), \Psi^{\#}(\mathcal{O}_{3n}),$

defined in section 4.2, is an isomorphism, and hence, together with Theorem (4.2), we have

Theorem (5.8): If $\overline{\overline{\Psi}}_*$: $H_{6n-4}(\mathcal{D}(n) - \mathcal{H}(n)/S^1) \longrightarrow H_{6n-4}(\mathbb{CP}^{3n-1})$ is a zero-homomorphism, then the degree homomorphism $\Psi_* : H_{6n}(\mathcal{W}(n), \Psi^*(\mathcal{O}_{3n})) \longrightarrow H_{6n}(P(3n), \mathcal{O}_{3n})$ is also a 0-map.

In Chapter 6, we shall show that $\mathcal{D}(n)$ has finite structure group. Thus, $\overline{\overline{\Psi}}_*$ in Theorem (5.8) is a 0-homomorphism, and hence, the degree homomorphism Ψ_* is also trivial.

Secondly, we consider the set $\mathcal{C} = \mathcal{W}_0(n) - \mathcal{D}(n)$. Then $\mathcal{W}_0(n)/\mathcal{C}$ is exactly the Thom space $T(\mathcal{D}(n))$ of the vector bundle $\mathcal{D}(n)$. Thus, we have the following exact sequence

$0 = H_{6n-2}(\mathcal{C}, \Psi^*(\mathcal{O}_{3n})) \longrightarrow H_{6n-2}(\mathcal{W}_0(n), \Psi^{\#}(\mathcal{O}_{3n})) \longrightarrow H_{6n-2}(T(\mathcal{D}(n)) \longrightarrow H_{6n-3}(\mathcal{C}, \Psi^{\#}(\mathcal{O}_{3n}))$.

By Thom isomorphism Theorem,

$$H_{6n-2}(T(\mathcal{D}(n)) \approx H_{2n-2}(\mathcal{H}(n)).$$

Therefore, we have

(5.9): $\Delta_* : H_{6n-2}(\mathcal{W}_0(n), \Psi^{\#}(\mathcal{O}_{3n})) \longrightarrow H_{2n-2}(\mathcal{H}(n))$ is an injective map.

With a straightforward, but tedious, computation of $H_*(\mathcal{H}(n))$, we can show that the image of Δ_* contains exactly the cycles from the weight system of Vassiliev invariant of order n. On the other hand, one can convince himself easily that a weight system ω will get cycles $\sum_\Gamma \omega(\Gamma)W_0(\Gamma)$ in $(\mathcal{W}_0(n), \Psi^\#(\mathcal{O}_{3n}))$ and $\sum_\Gamma \omega(\Gamma)W(\Gamma)$ in $(\mathcal{W}(n), \Psi^\#(\mathcal{O}_{3n}))$. And the degree homomorphism send it to 0, which is a non-zero multiple of Feynman integral over $\sum_\Gamma \omega(\Gamma)W(\Gamma)$. This concludes that the Feynman integral $\sum_\Gamma \frac{f_\Gamma}{|\Gamma|}\omega(\Gamma)$ is 0, for any weight system, and hence, $\sum_\Gamma \frac{f_\Gamma}{|\Gamma|}[\Gamma]$ is 0 in the algebra of chord diagram.

Remark: The computation of $H_*(\mathcal{H}(n))$ will appear in additional chapter (seven) or another article. Although it is lengthy, it has no problem anyway.

§6 Finite structure group

6.1 Finite structure group

Suppose Γ is a knot graph. $\overline{H}(\Gamma)$ is the set of all real function h on $V(\Gamma)$ such that h preserves the linear order of base points of $V(\Gamma)$ and h does not degenerates any edge in $V(\Gamma)$, precisely, for any two base points x and x' with $x < x'$, $h(x) < h(x')$, and for any edge $e = \{v, w\}$ in Γ, $h(v) \neq h(w)$. $\overline{D}(\Gamma) = \{(g, h) : h \in \overline{H}(\Gamma)$ and g is a complex-value function on $V(\Gamma)$ such that $g|_{V_0(\Gamma)} = 0\}$. $H(\Gamma)$ and $D(\Gamma)$ are the quotient spaces of $\overline{H}(\Gamma)$ and $\overline{D}(\Gamma)$ by all possible extended translation and dilation relations.

Suppose Γ has s inner vertices. Then $D(\Gamma)$ is an s-dimensional complex vector bundle over $H(\Gamma)$.

Assume Γ has k edges $e_1, e_2, \cdots, e_k$. Then, we have a fibrewise linear injective map $\Psi : D(\Gamma) \longrightarrow \mathbb{C}^k$, defined by

$$\Psi(g, h) = \left(\frac{g}{h}(e_1), \frac{g}{h}(e_2), \cdots, \frac{g}{h}(e_k)\right),$$

which depends on an order of edges $\left(\frac{g}{h}(e) = \frac{g(v)-g(w)}{h(v)-h(w)}, \text{ if } e = \{v, w\}\right)$. The E.T.D. relations are exactly the a priori relations having the same image under Ψ.

6.2 A trivialization of $D(\Gamma)$

Fix a linear order for the inner vertices in Γ, say, $y_1 < y_2 < \cdots < y_s$, for $y_1, y_2, \cdots, y_s$ in $V_1(\Gamma)$.

Suppose $h : V(\Gamma) \longrightarrow \mathbb{R}$ is a height function. For any edge $e = \{v, w\}$, let $|h|(e) = |h(v) - h(w)|$. A sequence of vertice $\eta = (v_0, v_1, \cdots, v_r)$ is said to be an arc connecting v_0 to v_r, if $\{v_{i-1}, v_i\}$, $i = 1, 2, \cdots, r$, are edges in Γ. Let $|h|(\eta) = \sum\limits_{i=1}^{r} |h|(v_{i-1}, v_i)$, it is said to be the h-length of the arc η. $|h|$ gives a

metric on Γ as follows:

$$v, w \in V(\Gamma),$$
$$d(h, v, w) = \min\{|h|(\eta) : \eta \text{ is arc connecting } v \text{ to } w\}.$$

For any $y_i, 1 \leq i \leq s$, let

$$V(y_i) = V_0(\Gamma) \cup \{y_{i+1}, y_{i+2}, \cdots, y_s\} \text{ and}$$
$$d(h, y_i) = \min\{d(h, v, y_i) : v \in V(y_i)\}.$$

Note: $d(h, y_i)$ is much dependent on the linear order.

Definition (6.1): For eeach i, $1 \leq i \leq s$,

(i) $g(h, y_i) : V(\Gamma) \longrightarrow \mathbb{C}$ is a ground function defined by: $g(h, y_i)(v) = \max\{0, d(h, y_i) - d(h, y_i, v)\}$, for any $v \in V(\Gamma)$.

(ii) $b(h, y_i)$ (or simply $b_i(h)$) is equal to $(g(h, y_i), h)$, it is in $D(\Gamma)_h$.

(iii) $\theta(h) : \mathbb{C}^s \longrightarrow \mathbb{C}^k$ is a linear map defined by: $\theta(h)(c_1, c_2, \cdots, c_s) = \Psi(c_1 b_1(h) + c_2 b_2(h) + \cdots + c_s b_s(h))$.

Proposition (6.2): Suppose every inner vertices are connecting to base points. Then

(i) $(b_1, b_2, \cdots, b_s)$ is a trivialization of the vector bundle $D(\Gamma)$ over $H(\Gamma)$.

(ii) $\theta : H(\Gamma) \longrightarrow \text{End } (\mathbb{C}^s, \mathbb{C}^k)$ sends $H(\Gamma)$ into a bounded set of End $(\mathbb{C}^s, \mathbb{C}^k)$.

(iii) For each $h \in H(\Gamma), \theta(h)$ is an injective linear map.

Proof:

(i) For $j > i$, $y_j \in V(y_i)$. $d(h, y_i) \leq d(h, y_j, y_i)$. Thus, $g(h, y_i)(y_j) = 0$. For any $v \neq w$ in $V(\Gamma)$, $d(h, v, w) > 0$, no matter that $h(v)$ is equal to $h(w)$ or not. $d(h, y_i) > 0$, for all $i = 1, 2, \cdots, s$. Thus $\{g(h, y_i), i = 1, 2, \cdots, s\}$ are linearly independent. This proves (i).

Note: If there is an inner vertice which is not connecting to any base point, the dimension of $D(\Gamma)_h$ is smaller than s.

(ii) $\Psi(b_i(h)) = \left(\frac{g(h,y_i)}{h}(e_r)\right)_{r=1}^{k}$.
For any edge $e = \{v, w\}$,
$|d(h, y_i, v) - d(h, y_i, w)| \leq d(h, v, w) = |h(w) - h(v)|$.
Thus,

$$\begin{aligned} \left|\frac{g(h, y_i)}{h}(e)\right| &= \frac{|g(h, y_i)(v) - g(h, y_i)(w)|}{|h(v) - h(w)|} \\ &\leq \frac{|g(h, y_i, v) - g(h, y_i, w)|}{|h(v) - h(w)|} \leq 1 \end{aligned}$$

and $|\Psi(b_i(h))| \leq \sqrt{k}$ in Euclidean norm. This prove (ii).

(iii) Assume $\theta(c_1, c_2, \cdots, c_s) = 0$ in $\mathbb{C}^k$. Let $g = c_1 gh, y_1) + c_2 g(h, y_2) + \cdots + c_s g(h, y_s)$.
Then $\frac{g}{h}(e) = 0$, for all $e \in E(\Gamma)$, that is $g(v) = g(w)$, for any $e = \{v, w\}$ in $E(\Gamma)$. By assumption, any inner vertice is connecting to a base point. Thus $g(v) = 0$, for all $v \in V(\Gamma)$. By (i), $c_1 = c_2 = \cdots = c_s = 0$. This proves (iii).

6.3 Boundary behavior of the trivialization

Suppose A is a subset of vertices in Γ. As in section 5.2, $Q(A(\Gamma)) = W_0(A(\Gamma))/$ S.E.T.D. relation, $Q(\Gamma; A) = W_0(\Gamma/A) \times Q(A(\Gamma))$. To restrict to the case that $Q(\Gamma; A)$ is a codimension 1 boundary, we assume that (i) $A(\Gamma)$ is connected, or (ii) A consists of two neighboring base points. But,

the second case is quite trivial, we may consider the first case only. Thus, $Q(A(\Gamma)) = W_0(A(\Gamma))$ and $Q(\Gamma; A) = W_0(\Gamma; A) \times W_0(A(\Gamma))$. Furthermore, $D(\Gamma; A) = D(\Gamma/A) \times D(A(\Gamma))$ and $H(\Gamma; A) = H(\Gamma/A) \times H(A(\Gamma))$.

6.3.1 Having base points

Suppose there is at least one base point in A.

Then, the set of inner vertices $V_1(\Gamma)$ splits into disjoint union of $V_1(A(\Gamma))$ and $V_1(\Gamma/A)$. Thus, the linear order on $V_1(\Gamma)$ gives linear orders on $V_1(A(\Gamma))$ and $V_1(\Gamma/A)$. For convenience, consider the following values for a height function h on Γ:

$\varepsilon(h) = \min\{|h(v) - h(w)|$: for all edge $e = \{v, w\}$ in $\Gamma\}$,

and $|h| = \max\{|h(v) - h(w)|$: for any two vertice v, w in $\Gamma\}$.

$\varepsilon(h)$ and $|h|$ are not invariant under T.D. relation, they are not a function on $H(\Gamma)$.

For any h_1 in $H(\Gamma/A)$ and h_2 in $H(A(\Gamma))$, let $h_\lambda : V(\Gamma) \longrightarrow \mathbb{R}$ defined by:

$$h_\lambda(v) = \begin{cases} h_1(v), & \text{if } v \text{ is not in } A, \\ \\ h_1(a) + \lambda\frac{\varepsilon(h_1)}{|h_2|}(h_2(v) - h_2(a_0)),, & \text{if } v \text{ is in } A, \end{cases}$$

where a denote the new base point corresponding to A, and a_0 is a fixed base point in A. When λ is sufficiently small, h_λ is a height function of Γ.

Let $b(h_\lambda, y_1)$, $b(h_\lambda, y_2)$, $\cdots$, $b(h_\lambda, y_s)$ denote the basis of $D(\Gamma)_{h_\lambda}$ constructed in section 6.2. The following results are straightforward.

Proposition (6.3):

(i) If $y \in V_1(\Gamma/A)$,

$$\lim_{\lambda \to 0} g(h_\lambda, y)(v) = \begin{cases} g(h_1, y)(v), & \text{for } v \notin A, \\ \\ g(h_1, y)(a), & \text{for } v \in A. \end{cases}$$

(ii) If $y \in A \cap V_1(\Gamma)$, $b(h_\lambda, y)|_{A(\Gamma)} = b(h_2, y)$, for all $\lambda > 0$.

(iii) If $y \in A \cap V_1(\Gamma)$, $\lim_{\lambda \to 0} g(h_\lambda, y)(v) = 0$, for $v \notin A$.

In general, if a parameter of height function $\{h_\lambda, \lambda > 0\}$ satisfying:

(i) $\lim_{\lambda \to 0} h_\lambda = h_1$ and

(ii) $\lim_{\lambda \to 0}(\varphi(\lambda) h_\lambda|_A + \mu(\lambda)) = h_2$, for some $\varphi(\lambda) > 0$ and $\mu(\lambda)$,

we also have the result

(i) in Proposition (6.3) and

(ii)' $\lim_{\lambda \to 0} b(h_\lambda, y)|_{A(\Gamma)} = b(h_2, y)$, for $y \in A \cap V_1(\Gamma) = V_1(A(\Gamma))$.

6.3.2 Inner vertices only

Suppose A contains inner vertices only, and y_r is the maximal point in A. It is convenience to think the new inner vertice of Γ/A corresponding to A is y_r. Thus, Γ/A has inner vertices $\overline{y}_r$ and the points in $V_1(\Gamma) - A$, and $A(\Gamma)$ has inner vertices the all points in A. ($\overline{y}_r$ is the new inner vertice in Γ/A) The linear order of A is just the restriction from that in $V_1(\Gamma)$ and the linear order of $V_1(\Gamma/A)$ is also the restriction to $\{y_r\} \cup (V_1(\Gamma) - A)$ and change the y_r to $\overline{y}_r$, that is, for $y_i \in V_1(\Gamma) - A$, $y_i < \overline{y}_r$, if and only if, $y_i < y_r$.

Similar to the above case, for $h_1 \in H(\Gamma/A)$ and $h_2 \in H(A(\Gamma))$, let h_λ be the height function on Γ defines by:

$$h_\lambda(v) = \begin{cases} h_1(v), & \text{if } v \in V(\Gamma) - A, \\ \\ h_1(\overline{y}_r) + \lambda \frac{\epsilon(h_1)}{|h_2|}(h_2(v) - h_2(y_r)), & \text{if } v \in A. \end{cases}$$

It is obvious that, when λ is sufficiently small ($\lambda > 0$), h_λ is a well-defined height function on Γ. Now, by the definition of the basis (b_i), we have

(i) $\{b(h_1, y)$, for $y \in V_1(\Gamma) - A$, and $b(h_1, \overline{y}_r)\}$,

(ii) $\{b(h_2, y)$, for $y \in A\}$ and

(iii) $\{b(h_\lambda, y), y \in V_1(\Gamma)\}$.

The first and the third are bases for $D(\Gamma/A)_{h_1}$ and $D(\Gamma)_{h_\lambda}$, obviously. But, the second is not a basis for $D(A(\Gamma))_{h_2}$. Let $s_1 = |A|$, the number of elements in A. By T.D. relation, $D(A(\Gamma))_{h_2}$ has dimension $s_1 - 1$.

By assumption $A(\Gamma)$ is connected, $d(h_2, y, y')$ is a well-defined non-negative number, for $y, y' \in A$. Thus, $d(h_2, y) = \min\{d(h_2, y, y') : y' > y\}$ is also a well-defined finite value, for $y \neq y_r$. Therefore, it is reasonable to consider the basis $\{b(h_2, y) : y \in A$ and $y \neq y_r\}$ as the trivialization of $D(A(\Gamma))$ over $H(A(\Gamma))$.

Proposition (6.4): Suppose A is a subset of $V_1(\Gamma)$ such that $A(\Gamma)$ is connected and $y_1 < y_2 < \cdots < y_{s_1}$ is the linear order of elements in A restrict from that in $V_1(\Gamma)$. Then $\{b(h_2, y_1), b(h_2, y_2), \cdots, b(h_2, y_{s_1 - 1})\}$, for $h_2 \in H(A(\Gamma))$, is a trivialization for $D(A(\Gamma))$ over $H(A(\Gamma))$.

Proposition (6.5): $\{h_\lambda, \lambda > 0\}$ defined above.

(i) If $y \in V_1(\Gamma/A)$ and $y \neq \overline{y}_r$,

$$\lim_{\lambda \to 0} g(h_\lambda, y)(v) = \begin{cases} g(h_1, y)(v), & \text{for } v \notin A, \\ g(h_1, y)(\overline{y}_r), & \text{for } v \in A. \end{cases}$$

(ii)

$$\lim_{\lambda \to 0} g(h_\lambda, y_r)(v) = \begin{cases} g(h_1, \overline{y}_r)(v), & \text{for } v \notin A, \\ g(h_1, \overline{y}_r)(\overline{y}_r), & \text{for } v \in A. \end{cases}$$

(iii) For $y \in A - \{y_r\}$,

$$b(h_\lambda, y)|_{A(\Gamma)} = b(h_2, y), \text{ for } \lambda \text{ sufficiently small,}$$

where the equality means that they are equivalent under T.D. relation.

(iv) For $y \in A - \{y_r\}$ and $v \notin A$,

$$\lim_{\lambda \to 0} h(h_\lambda, y)(v) = 0.$$

Proof. (i) Assume $y \in V_1(\Gamma/A)$ and $y \neq \overline{y}_r$. $d(h_\lambda, y) = \min\{d(h_\lambda, y, v) : v \in V_0(\Gamma) \text{ or } (v \in V_1(\Gamma) \text{ and } v > y \text{ in } V_1(\Gamma))\}$, it is smaller than or equal to $d'(h_\lambda, y) = \min\{d(h_\lambda, y, v) : v \in V_0(\Gamma) \text{ or } (v \in \{y_r\} \cup (V_1(\Gamma) - A) \text{ and } v > y \text{ in } V_1(\Gamma))\}$.

Claim: $\lim_{\lambda \to 0} d(h_\lambda, y) = \lim_{\lambda \to 0} d'(h_\lambda, y)$.

(Case 1): For all $v \in A$, $v < y$.

$$d(h_\lambda, y) = d'(h_\lambda, y), \quad \text{for all } \lambda.$$

(Case 2): There exists $v \in A$, $v > y$.

Then $y_r > y$. When λ approaches to 0, $d(h_\lambda, y, v)$ and $d(h_\lambda, y, y_r)$ have the same limist $(d(h_1, y, \overline{y}_r))$. Therefore, although $d'(h_\lambda, y)$ has less minimum set, it also has the same limist as $d(h_\lambda, y)$.
And, it is obvious that $d'(h_\lambda, y) = d(h_1, y)$. This implies the equality in (i) easily.

(ii) The set $V(y_r) = V_0(\Gamma) \cup \{v \in V_1(\Gamma), v > y_r\}$ is equal to $V(\overline{y}_r) = V_0(\Gamma/A) \cup \{v \in V_1(\Gamma/A),\ v > y_r\}$. Thus, $\lim_{\lambda \to 0} d(h_\lambda, y_r) = d(h_1, \overline{y}_r)$. This implies (ii).

(iii): For $y \in A$, $y \neq y_r$,

$$V(y) = V_0(\Gamma) \cup \{v \in V_1(\Gamma) : v > y\} \text{ for } \Gamma,$$

and

$$V'(y) = \{v \in A, v > y\}, \text{ for } A(\Gamma).$$

Now, compare $d(h_\lambda, y)$ with $d(h_\lambda|_A, y)$:

$$d(h_\lambda, y) = \min\{d(h_\lambda, y, v) : v \in V(y)\}$$
$$d(h_\lambda|_A, y) = \min\{d(h_\lambda, y, v) : v \in V'(y)\}$$

y_r is in $V(y)$, y_r is also in $V'(y)$. For any $v \in V(y) - V'(y)$, that is, $v \notin A$ and $v \in V(y)$, $d(h_\lambda, y, v) > d(h_\lambda, y, y_r)$, for λ sufficiently small. Thus, when λ is sufficiently small, $d(h_\lambda, y) = d(h_\lambda|_A, y)$. But, $d(h_\lambda|_A, y) = \lambda d(h_2, y)$. Thus,

$$(g(h_\lambda, y)(v), h_\lambda(v)) = \lambda(g(h_2, y)(v), h_2(v)) + (0, h_1(\overline{y}_r) - \lambda h_2(y_r)),$$

for all $v \in A$, that is, $b(h_\lambda, y)|_{A(\Gamma)}$ is equivalent to $b(h_2, y)$ under T.D. relation.

Remark (6.6): For $y \in A - \{y_r\}$, $g(h_\lambda, y)(v) = 0$, for $v \in V(\Gamma) - A$ and λ sufficiently small.

Proof: $d(h_\lambda, y, v) > d(h_\lambda, y) = \lambda d(h_2, y)$. Thus, $g(h_\lambda, y)(v) = \max\{ 0, d(h_\lambda, y) - d(h_\lambda, y, v)\} = 0$.

6.3.3 (Continued)

$\{h_\lambda, \lambda > 0\}$, as above, is a parameter of height function on Γ such that $\lim_{\lambda > 0} h_\lambda = h_1$ in $H(\Gamma/A)$ and $\lim_{\lambda \to 0}(h_\lambda|_{A(\Gamma)}) = h_2$ (in the sense that they are equivalence under T.D. relation) in $H(A(\Gamma))$. Part of limit behavior of $b(h_\lambda, y)$, $y \in V_1(\Gamma)$, is already studied. But, there are still something, we forget to study in section 6.3.1 and 6.3.2, that is, when $y \in V_1(\Gamma) - A$, the limit of the restriction of $b(h_\lambda, y)$ to $A(\Gamma)$. Actually, the limit does not exist. Thus, we should modify $b(h_\lambda, y)$ such that its restriction to $A(\Gamma)$ will approach to zero.

An easy approach is the following:

For $y \in V_1(\Gamma) - A$, let $\overline{g}(h_\lambda, y)$ be a ground function such that, for any $e = \{v, w\}$,

$$\overline{g}(h_\lambda, y)(v) - \overline{g}(h_\lambda, y)(w)$$

$$= \begin{cases} g(h_\lambda, y)(v) - g(h_\lambda, y)(w), & \text{for } e \notin E(A(\Gamma)), \\ \\ \lambda(g(h_\lambda, y)(v) - g(h_\lambda, y)(w)), & \text{for } e \in E(A(\Gamma)). \end{cases}$$

(note: $|g(h_\lambda, y)(v) - g(h_\lambda, y)(w)| \leq |h|(e)$)

When λ is small, we switch $g(h_\lambda, y)$ to $\overline{g}(h_\lambda, y)$, continuously, and we get a new basis $\{\overline{b}(h_\lambda, y), y \in A \cap V_1(\Gamma)\}$. Then, Proposition (6.3) and (6.5) also hold for this new basis. For example:

$$\lim_{\lambda \to 0} \overline{g}(h_\lambda, y)(v) = g(h_1, y)(v),$$

for $y \in V_1(\Gamma/A)$ and $v \notin A$, in (6.3). Moreover, we also have

Proposition (6.7):

(i) For any $y \in V_1(\Gamma) - A$,

$$\lim_{\lambda \to 0} \overline{b}(h_\lambda, y)|_{A(\Gamma)} = 0.$$

(ii) If $A \subset V_1(\Gamma)$ and y_r is the maximal element in A, then

$$\lim_{\lambda \to 0} b(h_\lambda, y_r)|_{A(\Gamma)} = b(h_2, y_r).$$

Remark: In the case (ii) of the above proposition, we may define $\overline{b}(h_\lambda, y_r)$ similarly such that $\lim\limits_{\lambda \to 0} \overline{b}(h_\lambda, y_r)|_{A(\Gamma)} = 0$ as in (i).

Thus, by (6.3), (6.5) and (6.7), we find that the limit of the new basis $\{b(h_\lambda, y)\}$ is completely determined by the two bases at the limit (h_1, h_2) of $\{h_\lambda\}$, in $H(\Gamma/A) \times H(A(\Gamma))$.

The above modification of basis is done on a collaring neighborhood of $H(\Gamma; A)$ in $H(\Gamma)$. Thus, we need a non-overlapping collaring neighborhood for each codimension-1 boundary $H(\Gamma; A)$. We can believe that the space is good enough to do so. If one does not believe it, one could try an alternating method: a more general modification on all $g(h, y)$ in the following way:

(i) For any two functions $g_1, g_2 : V(\Gamma) \longrightarrow \mathbb{C}$ and $e = \{v, w\}$, $\frac{g_1}{g_2}(e) = \frac{g_1(v)-g_1(w)}{g_2(v)-g_2(w)}$.

(ii) $E(y) = \{e = \{y, v\}$, e is an edge in $\Gamma\}$.

(iii) $\overline{g}(h, y) : V(\Gamma) \longrightarrow \mathbb{C}$ is a ground function on Γ satisfying

$$\frac{\overline{g}(h,y)}{g(h,y)}(e) = \begin{cases} \frac{5|h|(e)}{\delta(y)}, & \text{if } |h|(e) \leq \frac{1}{5}\delta(y), \\ \\ 1, & \text{if otherwise,} \end{cases}$$

where $|h|(e) = |h(v) - h(w)|$ $(e = \{v, w\})$ and $\delta(y) = \min\{|h|(e) : e \in E(y)\}$.

(That is, we shrink the effect of $|h|(e)$ on $g(h, y)$, when $|h|(e)$ is much smaller than $\delta(y)$, the h-lengths of edges connecting to y.)

Note: $\frac{\overline{g}}{g}(e) = 1$ means that $\overline{g}(v) - \overline{g}(w) = g(v) - g(w)$, whenever $g(v) = g(w)$ or not.

Using the modified basis $\{b(h, y) = (\overline{g}(h, y), h)\}$, we also have the result (6.3), (6.5) and (6.7), except the result (ii) in (6.5) becoming that

$$\text{(ii)}' \ \lim_{\lambda \to 0} \overline{g}(h_\lambda, y_r)(v) = \begin{cases} g(h_1, \overline{y}_r)(v), \text{for } v \notin A \\ \\ g(h_1, \overline{y}_r)(\overline{y}_r), \text{for } v \in A \end{cases}$$

(its limit returns to the original one).

6.4 Isotopy of trivializations

Changing the linear order of inner vertices, we get different trivializations of $D(\Gamma)$. We shall show that they are all isotopic.

As in section 6.2, $y_1 < y_2 < \cdots < y_s$ are the inner vertices with linear order (in Γ). $V(y_i) = V_0(\Gamma) \cup \{y_{i+1}, y_{i+2}, \cdots, y_s\}$, $i = 1, 2, \cdots, s$, and $d(h, y_i) = \min\{d(h, v, y_i) : v \in V(y_i)\}$.

Now, assume r is an integer, $1 \leq r \leq s-1$, and consider the linear order $y_1 < y_2 < \cdots < y_{r-1} < y_{r+1} < y_r < y_{r+2} < \cdots < y_s$, which switches the positions of y_r and y_{r+1}. Then, we should consider

$$\begin{aligned}
&V'(y_r) = V_0(\Gamma) \cup \{y_{r+2}, y_{r+3}, \cdots, y_s\}, \\
&V'(y_{r+1}) = V_0(\Gamma) \cup \{y_r, y_{r+2}, y_{r+3}, \cdots, y_s\}, \\
&d'(h, y_r) = \min\{d(h, v, y_r) : v \in V'(y_r)\}, \\
&d'(h, y_{r+1}) = \min\{d(h, v, y_{r+1}) : v \in V'(y_{r+1})\}.
\end{aligned}$$

for convenience, let $V'(y_i) = V(y_i)$ and $d'(h, y_i) = d(h, y_i)$, for $i \neq r$ and $r+1$. Furthermore, let $g'(h, y_i)$ be the ground function defined by $g'(h, y_i)(v) = \max\{0, d'(h, y_i) - d(h, v, y_i)\}$. And $b'(h, y_i) = (g'(h, y_i), h)$, $i = 1, 2, \cdots, s$. Then, $\{b'(h, y_i),\ i = 1, 2, \cdots, s\}$ is also a trivialization on $D(\Gamma)$ over $H(\Gamma)$. We shall show that $\{b(h, y_i)\}$ and $\{b'(h, y_i)\}$ can be homotopic to a trivialization $\{\overline{b}(h, y_i)\}$ through parameters of trivialization $\{b_t(h, y_i)\}$ and $\{b'_t(h, y_i)\}$, respectively.

Consider the following, as above:

(i) $\overline{V}(y_r) = \overline{V}(y_{r+1}) = V'(y_r) \cap V'(y_{r+1})$, that is, $V'(y_r)$.

(ii) $\overline{V}(y_i) = V(y_i)$, for $i \neq r$, $r+1$.

(iii) $\overline{d}(h, y_i) = \min\{d(h, v, y_i) : v \in \overline{V}(y_i)\}$.

(iv) $\overline{g}(h, y_i)(v) = \max\{0, \overline{d}(h, y_i) - d(h, v, y_i)\}$

(v) $\overline{b}(h, y_i) = (\overline{g}(h, y_i), h)$, $i = 1, 2, \cdots, s$.

It is easy to see that $\overline{d}(h, y_i) \geq d(h, y_i)$ and $\overline{d}(h, y_i) \geq d'(h, y_i)$, $i = 1, 2, \cdots, s$.
Let $d_t(h, y_i) = td(h, y_i) + (1-t)\overline{d}(h, y_i)$, for $0 \leq t \leq 1$.
Also, $g_t(h, y_i)(v) = \max\{0, d_t(h, y_i) - d(h, v, y_i)\}$ and $b_t(h, y_i) = (g_t(h, y_i), h)$, $i = 1, 2, \cdots, s$. Similarly, for $d'_t(h, y_i)$, $g'_t(h, y_i)$ and $b'_t(h, y_i)$.

Proposition (6.8): For any t, $0 \leq t \leq 1$, and $h \in H(\Gamma)$, both $\{b_t(h, y_i)\}_{i=1}^s$ and $\{b'_t(h, y_i)\}_{i=1}^s$ are bases for $D(\Gamma)_h$. Thus, the two trivialization $\{b(h, y_i)\}$ and $\{b'(h, y_i)\}$ are isotopic through the parameter of trivializations $\{b_t(h, y_i), i = 1, 2, \cdots, s\}$ and $\{b'_t(h, y_i), i = 1, 2, \cdots, s\}$.

Proof: It is easy to show that, for $0 \leq t \leq 1$, $g_t(h, y_i)$, $i = 1, 2, \cdots, s$, are linearly independent (Similarly, for $\{g'_t(h, y_i)\}$). Because $\overline{d}(h, y_i) \geq d_t(h, y_i)$,

(i) $g_t(h, y_i)(y_j) = 0$, for $i \leq r - 1$ and $j > i$,

(ii) $g_t(h, y_r)(y_j) = 0$, for $j \geq r + 2$,

(iii) $g_t(h, y_{r+1})(y_j) = 0$, for $j \geq r + 2$,

(iv) $g_t(h, y_i)(y_j) = 0$, for $i \geq r + 2$ and $j > i$.

Moreover, $g_t(h, y_i)(y_i) > g_t(h, y_i)(y_j) \geq 0$, for any $1 \leq j \neq i \leq s$.

Consider the matrix $G = (G_{i,j})_{\substack{1 \leq i \leq s \\ 1 \leq j \leq s}}$, $G_{i,j} = g_t(h, y_i)(y_j)$. The above conditions implies that the determinant of G is greater than 0. In fact,

$$\begin{aligned} \det(G) \quad &= G_{11}G_{22} \cdots G_{ss} - G_{11}G_{22} \cdots G_{r-1,r-1}G_{r,r+1}G_{r+1,r}G_{r+2,r+2} \cdots G_{s,s} \\ &= G_{11}G_{22} \cdots G_{ss} \cdot (G_{r,r}G_{r+1,r+1})^{-1} \cdot (G_{r,r} \cdot G_{r+1,r+1} - G_{r,r+1} \cdot G_{r+1,r}). \end{aligned}$$

$G_{r,r} > G_{r,r+1}$ and $G_{r+1,r+1} > G_{r+1,r}$. Thus, $\det(G) > 0$. This proves this proposition.

Therefore, there is no real difference to choose any linear order for $V_1(\Gamma)$ to construct the trivialization.

6.5 Transition maps of vector bundle $\mathcal{D}(n)$

Because we have trivializations $(b_1, b_2, \cdots, b_s)$ for every $D(\Gamma)$, the transition maps are given by all the identifications stated in section 1.3, or in section 5.3 (vector bundle form).

6.5.1 Identification of type 0

This is already absorbed by the special extended translation and dilation relation (see section 5.2).

6.5.2 Identification of type I

Suppose A is a subset of $V(\Gamma)$ and $A(\Gamma)$ has a univalent inner vertice v.

Assume $e = \{v, v_1\}$ is the unique edge in $A(\Gamma)$, containing v as the endpoint.

By the result of section 6.4, we may assume that v is the minimum element in $V_1(A(\Gamma))$. For convenience, let $y_1 = v < y_2 < \cdots < y_r$ denote the inner vertices in $A(\Gamma)$. Then, for any height function h on $A(\Gamma)$, $g(h, y_1)(y_1) = |h|(e)$, that is, $|h(v) - h(v_1)|$, and $g(h, y_1)(y_j) = 0$, for $j \neq 1$.

(6.9): $\tau_1' : D(A(\Gamma)) \longrightarrow D(A(\Gamma))$, defined in the method (ii) of section 5.3.4, satisfies the following:

(i) $\tau_1'(g(h, y_1), h)$ is exactly equal to $(g(\tau_1'(h), y_1), \tau_1'(h))$.

(ii) For $i \geq 2$ and $j \geq 2$, $\tau_1'(g(h, y_i))(y_j) = g(h, y_i)(y_j) = g(\tau_1'(h), y_i)(y_j)$.

Proof of (6.9): $\tau_1'(h)(w) = h(w)$, for $w \neq y_1 = v$. Thus,

$$g(\tau_1'(h), y_1)(y_1) = |\tau_1'(h)(v) - \tau_1'(h)(v_1)| = 2||h_1||$$

and $g(\tau_1'(h), y_1)(w) = 0$, for $w \neq y_1$.

$$\begin{aligned}
&\tau_1'(g(h, y_1))(y_1) \\
&= g(h, y_1)(v_1) + 2||h_1|| \frac{g(h, y_1)(v) - g(h, y_1)(v_1)}{|h(v) - h(v_1)|} \\
&= 2||h_1||
\end{aligned}$$

and $\tau_1'(g(h, y_1))(w) = g(h, y_1)(w) = 0$, for $w \neq y_1$. This proves (i).

The proof of (ii) is straightforward.

Thus, the restriction of τ_1' to $D(A(\Gamma))_h$, : $D(A(\Gamma))_h \longrightarrow D(A(\Gamma))_{\tau_1'(h)}$, send b_1 to b_1 and b_i to $b_i + \rho_i b_1$, for $i \geq 2$, where ρ_i is a real number depending on h. By the homotopy property of vector bundle, we may change τ_1' to a homotopy one τ_1'', $\tau_1''(b_i) = b_i$ for all i.

Therefore, the transition map for Identification of type I is the identity map.

6.5.3 Identification of type II

Suppose A is a subset of $V(\Gamma)$ and $A(\Gamma)$ has a bivalent inner vertice v.

Assume $e_1 = \{v, w_1\}$ and $e_2 = \{v, w_2\}$ are the two edges connecting to v.

As above, we may assume that v is the minimum element in $V_1(A(\Gamma))$ and $y_1 = v < y_2 < \cdots < y_r$ are the inner vertices in $A(\Gamma)$. Then, for any height function of $A(\Gamma)$,

$$g(h, y_1)(y_1) = \min\{|h|(e_1), |h|(e_2)\}$$

and $g(h, y_1)(y_i) = 0$, $i > 1$.

As the notations in section 5.3.1, we have

(6.10):

(i) $\tau_2'(g(h, y_1)) = -g(\tau_2'(h), y_1)$.

(ii) For $i \geq 2$ and $j \geq 2$,

$$\begin{aligned} \tau_2'(g(h, y_i))(y_j) \quad &= g(h, y_i)(y_j) \\ &= g(\tau_2'(h), y_i)(y_j). \end{aligned}$$

Proof of (6.10):

$$\tau_2'(g(h, y_1))(y_1) \quad = g(h, y_1)(w_1) + g(h, y_1)(w_2) - g(h, y_1)(v)$$

$$
\begin{aligned}
&= -g(h, y_1)(y_1) \\
&= -\min\{|h|(e_2), |h|(e_1)\}. \\
g(\tau_2'(h), y_1)(y_1) &= \min\{|\tau_2'(h)|(e_1), |\tau_2'(h)|(e_2)\} \\
&= \min\{|h|(e_2), |h|(e_1)\}.
\end{aligned}
$$

This proves (i).

To prove (ii), choose arcs η_1 and η_2 from y_i to w_1 and w_2, respectively, such that η_1 minimizes the h-length from y_i to w_1 and η_2 minimizes the h-length from y_i to w_2.

(Case 1) y_1 is on η_1.

Then w_2 is also on η_1 and η_2 is a subarc of η_1.
Thus, $g(h, y_i)(w_2) \geq g(h, y_i)(y_1) \geq g(h, y_i)(w_1)$.
If $d(h, y_i) \leq d(h, w_2, y_i)$, then $g(h, y_i)(w_2) = g(h_1, y_i)(y_1) = g(h, y_i)(w_1) = 0$.
If $d(h, w_2, y_i) \leq d(h, y_i) \leq d(h, w_1, y_i)$, then
$g(h, y_i)(w_2) = d(h, y_i) - d(h, w_2, y_i)$ and $g(h, y_i)(w_1) = 0$.
If $d(h, w_1, y_i) \leq d(h, y_i)$, then

$g(h, y_i)(w_1) = d(h, y_i) - d(h, w_1, y_i)$ and $g(h, y_i)(w_2) = d(h, y_i) - d(h, w_2, y_i)$.
By assumption, $i \geq 2$, $y_i > y_1$, we have $d(h, y_i) = d(\tau_2'(h), y_i)$.
Thus,$g(h, y_i)(w_1) = g(\tau_2'(h), y_i)(w_1)$ and $g(h, y_i)(w_2) = g(\tau_2'(h), y_i)(w_2)$.

(Case 2) y_1 is on η_2.

Then w_1 is also η_2 and η_1 is a subarc of η_2. And we can prove that

$$
g(h, y_i)(w_j) = g(\tau_2'(h), y_i)(w_j), \;\; j = 1, 2,
$$

as in (case 1).

(Case 3) y_1 is not on $\eta_1 \cup \eta_2$.

Then

$$
d(h, y_i, w_1) = d(\tau_2'(h), y_i, w_1)
$$

$$\text{and} \qquad d(h, y_i, w_2) = d(\tau_2'(h), y_i, w_2).$$

Thus, we also have the equalities

$$g(h, y_i)(w_j) = g(\tau_2'(h), y_i)(w_j), j = 1, 2.$$

The proof of other results is similar and is omitted.

What is the difference of $\tau_2' g(h, y_i))(y_1)$ and $g(\tau_2'(h), y_i)(y_1)$, for $i \geq 2$?

(6.11): For $i \geq 2$,

$$(\tau_2'(g((h, y_i))(y_1) - g(\tau_2'(h), y_i)(y_1)| \leq 2 \min\{|h|(e_1), |h|(e_2)\}.$$

Proof:
$|\tau_2'(g((h, y_i))(y_1) - \tau_2'(g(h, y_i)(w_1)| = |g(h, y_i)(w_2) - g(h, y_i)(y_1)| \leq |h|(e_2)$.
$|g(\tau_2'(h), y_i)(y_1) - g(\tau_2'(h), y_i)(w_1)| \leq |\tau_2'(h)|(e_1) = |h|(e_2)$.
By (6.10), $\tau_2'(g(h, y_i)))(w_1) = g(\tau_2'(h), y_i)(w_1)$.

Thus, $|\tau_2'(g(h, y_i))(y_1) - g(\tau_2'(h), y_i)(y_1)| \leq 2|h|(e_2)$.
Similarly,

$$|\tau_2'(g(h, y_i))(y_1) - g(\tau_2'(h), y_i)(y_1)| \leq 2|h|(e_1).$$

This proves (6.11).

Now, consider the restriction of τ_2' to $D(A(\Gamma))_h, : D(A(\Gamma))_h \longrightarrow D(A(\Gamma))_{\tau_2'(h)}$, it sends $b(h, y_1)$ to $-b(\tau_2'(h), y_1)$ and sends $b(h, y_i)$ to $b(\tau_2'(h), y_i) + \rho_i' b(\tau_2'(h), y_1)$, where ρ_i' is a real number depending on h and $|\rho_i'| \leq 2$.

Thus, we may assume that the identification map τ_2 has the following form: $\tau_2(b_1) = -b_1$ and $\tau_2(b_i) = b_i$, for $i > 1$, without changing the isomorphism class of vector bundle $\mathcal{D}(n)$.

6.5.4 Identification of type III

Suppose $A = \{v, w\}$ is an edge of Γ and v is an inner vertice. Assume v is the smallest inner vertice in A (if w is a base point, then the assumption holds automatically).

As the notations in section 5.3.2,

$$\tau_3(b(h,v)) = \frac{|h(v)-h(w)|}{h(v)-h(w)} = \pm 1 \text{ in } \mathbb{C},$$

for any height function h on $A(\Gamma)$.

Let $\{b(\cdot, y_i), y_i \neq v\}$ denote the trivialization for $D(\Gamma/A)$ and δ_1 the standard basis $\{1\}$ in $\mathbb{C}$. Then $\tau_3 : D(\Gamma; A) \longrightarrow D(\Gamma/A)\times\mathbb{C}$ has the following form:

$$\begin{aligned} \tau_3(b(\cdot, y_i)) &= b(\cdot, y_i), \text{ for } y_i \neq v, \\ \tau_3(b(\cdot, y_i)) &= \pm\delta_1. \end{aligned}$$

6.5.5 Identification of type IV

In this case, $A = \{\, v, w \,\}$ is not an edge in Γ, the identification map is nothing but the identity map.

6.5.6 Conclusion

Without changing the isomorphism class of $\mathcal{D}(n)$, the transition maps for the trivialization $(b_1, b_2, \cdots, b_s, \delta_1, \cdots, \delta_{3n-k})$ in $D(\Gamma)\times\mathbb{C}^{3n-k}\times\Sigma_{3n}\cdot\Psi$ has only two kinds:

(i) one is the permutation of basis.

(ii) One is the composition of permutation and θ_i, where $\theta_i(b_i) = -b_i$ and $\theta_i(b_j) = b_j$, for $j \neq i$.

Thus, $\mathcal{D}(n)$ has finite structure group.

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